Bulgaria National Olympiad 2008

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Day 1 May 17th

1 Let ABC be an acute triangle and CL be the angle bisector of $\angle ACB$. The point P lies on the segment CL such that $\angle APB = \pi - \frac{1}{2}\angle ACB$. Let k_1 and k_2 be the circumcircles of the triangles APC and BPC. $BP \cap k_1 = Q, AP \cap k_2 = R$. The tangents to k_1 at Q and k_2 at B intersect at S and the tangents to k_1 at A and k_2 at R intersect at T. Prove that $AS = BT$.

2 Is it possible to find 2008 infinite arithmetical progressions such that there exist finitely many positive integers not in any of these progressions, no two progressions intersect and each progression contains a prime number bigger than 2008?

3 Let $n \in \mathbb{N}$ and $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq \pi$ and $b_1, b_2, \ldots, b_n$ are real numbers for which the following inequality is satisfied :

$$\left|\sum_{i=1}^{n} b_i \cos(ka_i)\right| < \frac{1}{k}$$

for all $k \in \mathbb{N}$. Prove that $b_1 = b_2 = \ldots = b_n = 0$.

Day 2 May 18th

1 Find the smallest natural number k for which there exists natural numbers m and n such that $1324 + 279m + 5^n$ is k-th power of some natural number.

2 Let n be a fixed natural number. Find all natural numbers m for which

$$\frac{1}{a^n} + \frac{1}{b^n} \geq a^m + b^m$$

is satisfied for every two positive numbers a and b with sum equal to 2.

3 Let M be the set of the integer numbers from the range $[-n, n]$. The subset P of M is called a *base subset* if every number from M can be expressed as a sum of some different numbers from P. Find the smallest natural number k such that every k numbers that belongs to M form a base subset.