Bulgaria National Olympiad 2010

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Day 1

1 A table 2×2010 is divided to unit cells. Ivan and Peter are playing the following game. Ivan starts, and puts horizontal 2×1 domino that covers exactly two unit table cells. Then Peter puts vertical 1×2 domino that covers exactly two unit table cells. Then Ivan puts horizontal domino. Then Peter puts vertical domino, etc. The person who cannot put his domino will lose the game. Find who have winning strategy.

2 Each of two different lines parallel to the the axis Ox have exactly two common points on the graph of the function $f(x) = x^3 + ax^2 + bx + c$. Let ℓ_1 and ℓ_2 be two lines parallel to Ox axis which meet the graph of f in points K_1, K_2 and K_3, K_4, respectively. Prove that the quadrilateral formed by K_1, K_2, K_3 and K_4 is a rhombus if and only if its area is equal to 6 units.

3 Let $a_0, a_1, \ldots, a_9$ and $b_1, b_2, \ldots, b_9$ be positive integers such that $a_9 < b_9$ and $a_k \neq b_k, 1 \leq k \leq 8$. In a cash dispenser/automated teller machine/ATM there are $n \geq a_9$ levs (Bulgarian national currency) and for each $1 \leq i \leq 9$ we can take a_i levs from the ATM (if in the bank there are at least a_i levs). Immediately after that action the bank puts b_i levs in the ATM or we take a_0 levs. If we take a_0 levs from the ATM the bank doesnt put any money in the ATM. Find all possible positive integer values of n such that after finite number of takings money from the ATM there will be no money in it.

Day 2

1 Does there exist a number $n = \overline{a_1a_2a_3a_4a_5a_6}$ such that $\overline{a_1a_2a_3} + 4 = \overline{a_4a_5a_6}$ (all bases are 10) and $n = a^k$ for some positive integers a, k with $k \geq 3$?

2 Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(1) = 1$ and

$$f(n) = n - f(f(n-1)), \quad \forall n \geq 2.$$

Prove that $f(n + f(n)) = n$ for each positive integer n.

3 Let k be the circumference of the triangle ABC. The point D is an arbitrary point on the segment AB. Let I and J be the centers of the circles which are tangent to the side AB, the segment CD and the circle k. We know that the points A, B, I and J are concyclic. The excircle of the triangle ABC is tangent to the side AB in the point M. Prove that $M \equiv D$.