Bulgaria National Olympiad 2011

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Day 1

1 Prove whether or not there exist natural numbers n, k where $1 \le k \le n-2$ such that

$$\binom{n}{k}^2 + \binom{n}{k+1}^2 = \binom{n}{k+2}^4$$

2 Let $f_1(x)$ be a polynomial of degree 2 with the leading coefficient positive and $f_{n+1}(x) = f_1(f_n(x))$ for $n \ge 1$. Prove that if the equation $f_2(x) = 0$ has four different non-positive real roots, then for arbitrary n then $f_n(x)$ has 2^n different real roots.

3 Triangle ABC and a function $f : \mathbb{R}^+ \to \mathbb{R}$ have the following property: for every line segment DE from the interior of the triangle with midpoint M, the inequality $f(d(D)) + f(d(E)) \le 2f(d(M))$, where $d(X)$ is the distance from point X to the nearest side of the triangle (X is in the interior of $\triangle ABC$). Prove that for each line segment PQ and each point interior point N the inequality $|QN|f(d(P)) + |PN|f(d(Q)) \le |PQ|f(d(N))$ holds.

Day 2

1 Point O is inside $\triangle ABC$. The feet of perpendicular from O to BC, CA, AB are D, E, F. Perpendiculars from A and B respectively to EF and FD meet at P. Let H be the foot of perpendicular from P to AB. Prove that D, E, F, H are concyclic.

2 For each natural number a we denote $\tau(a)$ and $\phi(a)$ the number of natural numbers dividing a and the number of natural numbers less than a that are relatively prime to a. Find all natural numbers n for which n has exactly two different prime divisors and n satisfies $\tau(\phi(n)) = \phi(\tau(n))$.

3 In the interior of the convex 2011-gon are 2011 points, such that no three among the given 4022 points (the interior points and the vertices) are collinear. The points are coloured one of two different colours and a colouring is called "good" if some of the points can be joined in such a way that the following conditions are satisfied:
1) Each segment joins two points of the same colour.
2) None of the line segments intersect.
3) For any two points of the same colour there exists a path of segments connecting them.
Find the number of "good" colourings.