Flanders Math Olympiad 2006

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by Kurt Gdel

1 (a) Solve for $\theta \in \mathbb{R}$: $\cos(4\theta) = \cos(3\theta)$

(b) $\cos\left(\frac{2\pi}{7}\right)$, $\cos\left(\frac{4\pi}{7}\right)$ and $\cos\left(\frac{6\pi}{7}\right)$ are the roots of an equation of the form $ax^3 + bx^2 + cx + d = 0$ where a, b, c, d are integers. Determine a, b, c and d.

2 Let $\triangle ABC$ be an equilateral triangle and let P be a point on $[AB]$. Q is the point on BC such that PQ is perpendicular to AB. R is the point on AC such that QR is perpendicular to BC. And S is the point on AB such that RS is perpendicular to AC. Q' is the point on BC such that PQ' is perpendicular to BC. R' is the point on AC such that $Q'R'$ is perpendicular to AC. And S' is the point on AB such that $R'S'$ is perpendicular to AB.
Determine $\frac{|PB|}{|AB|}$ if $S = S'$.

3 Elfs and trolls are seated at a round table, 60 creatures in total. Trolls always lie, and all elfs always speak the truth, except when they make a little mistake.
Everybody claims to sit between an elf and a troll, but exactly two elfs made a mistake! How many trolls are there at this table?

4 Find all functions $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}$ such that

$$f(x) + f\left(\frac{1}{1 - x}\right) = 1 + \frac{1}{x(1 - x)}.$$