South africa National Olympiad 1995

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–	Part A
1	Prove that there are no integers m and n such that $$19m^2+95mn+2000n^2=1995.$$
2	ABC is a triangle with $\hat{A}<\hat{C}$, and D is the point on BC such that $B\hat{A}D=A\hat{C}B$. The perpendicular bisectors of AD and AC intersect in the point E. Prove that $B\hat{A}E=90^\circ$.
3	Suppose that $a_1,a_2,\dots,a_n$ are the numbers $1,2,3,\dots,n$ but written in any order. Prove that $$(a_1-1)^2+(a_2-2)^2+\cdots+(a_n-n)^2$$ is always even.
4	Three circles, with radii p, q and r and centres A, B and C respectively, touch one another externally at points D, E and F. Prove that the ratio of the areas of $\triangle DEF$ and $\triangle ABC$ equals $$\frac{2pqr}{(p+q)(q+r)(r+p)}.$$
–	Part B
1	The convex quadrilateral $ABCD$ has area 1, and AB is produced to E, BC to F, CD to G and DA to H, such that $AB=BE$, $BC=CF$, $CD=DG$ and $DA=AH$. Find the area of the quadrilateral $EFGH$.
2	Find all pairs (m,n) of natural numbers with $m<n$ such that m^2+1 is a multiple of n and n^2+1 is a multiple of m.
3	The circumcircle of $\triangle ABC$ has radius 1 and centre O and P is a point inside the triangle such that $OP=x$. Prove that $$AP\cdot BP\cdot CP\le(1+x)^2(1-x),$$ with equality if and only if $P=O$.