

**South africa National Olympiad 1997**

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- 1 From an initial triangle  $\Delta A_0 B_0 C_0$ , a sequence of triangles  $\Delta A_1 B_1 C_1, \Delta A_2 B_2 C_2, \dots$  is formed such that, at each stage,  $A_{k+1}, B_{k+1}$  and  $C_{k+1}$  are the points where the incircle of  $\Delta A_k B_k C_k$  touches the sides  $B_k C_k, C_k A_k$  and  $A_k B_k$  respectively.

(a) Express  $\angle A_{k+1} B_{k+1} C_{k+1}$  in terms of  $\angle A_k B_k C_k$ .

(b) Deduce that, as  $k$  increases,  $\angle A_k B_k C_k$  tends to  $60^\circ$ .

- 2 Find all natural numbers with the property that, when the first digit is moved to the end, the resulting number is  $\frac{7}{2}$  times the original one.

- 3 Find all solutions  $x, y \in \mathbb{Z}, x, y \geq 0$ , to the equation

$$1 + 3^x = 2^y.$$

- 4 Find all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  which satisfy

$$f(m + f(n)) = f(m) + n$$

for all  $m, n \in \mathbb{Z}$ .

- 5 A circle and a point  $P$  higher than the circle lie in the same vertical plane. A particle moves along a straight line under gravity from  $P$  to a point  $Q$  on the circle. Given that the distance travelled from  $P$  in time  $t$  is equal to  $\frac{1}{2}gt^2 \sin \alpha$ , where  $\alpha$  is the angle of inclination of the line  $PQ$  to the horizontal, give a geometrical characterization of the point  $Q$  for which the time taken from  $P$  to  $Q$  is a minimum.

- 6 Six points are connected in pairs by lines, each of which is either red or blue. Every pair of points is joined. Determine whether there must be a closed path having four sides all of the same colour. (A path is closed if it begins and ends at the same point.)