South africa National Olympiad 1999

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by Arne

1 How many non-congruent triangles with integer sides and perimeter 1999 can be constructed?

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2 $A,\ B,\ C$ and D are points on a given straight line, in that order. Show how to construct a square $PQRS$, with all of $P,\ Q,\ R$ and S on the same side of AD, such that $A,\ B,\ C$ and D lie on $PQ,\ SR,\ QR$ and PS produced respectively.

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3 The bisector of $\angle BAD$ in the parallellogram $ABCD$ intersects the lines BC and CD at the points K and L respectively. Prove that the centre of the circle passing through the points $C,\ K$ and L lies on the circle passing through the points $B,\ C$ and D.

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4 The sequence $L_1,\ L_2,\ L_3,\ \dots$ is defined by

$$L_1 = 1,\ \ L_2 = 3,\ \ L_n = L_{n-1} + L_{n-2} \text{ for } n > 2.$$

Prove that $L_p - 1$ is divisible by p if p is prime.

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5 Let S be the set of all rational numbers whose denominators are powers of 3. Let a, b and c be given non-zero real numbers. Determine all real-valued functions f that are defined for $x \in S$, satisfy

$$f(x) = af(3x) + bf(3x-1) + cf(3x-2) \text{ if } 0 \le x \le 1,$$

and are zero elsewhere.

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6 You are at a point (a, b) and you need to reach another point (c, d). Both points are below the line $x = y$ and have integer coordinates. You can move in steps of length 1, either upwards of to the right, but you may not move to a point on the line $x = y$. How many different paths are there?

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