South africa National Olympiad 2000

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by Arne

1 A number x_n of the form 10101...1 has n ones. Find all n such that x_n is prime.

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2 Solve for x, given $36x^4 + 36x^3 - 7x^2 - 6x + 1 = 0$.

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3 Let $c \geq 1$ be an integer, and define the sequence $a_1,\ a_2,\ a_3,\ \ldots$ by

$$a_1 = 2,$$
$$a_{n+1} = ca_n + \sqrt{\left(c^2 - 1\right)\left(a_n^2 - 4\right)} \text{ for } n = 1, 2, 3, \ldots .$$

Prove that a_n is an integer for all n.

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4 $ABCD$ is a square of side 1. P and Q are points on AB and BC such that $\widehat{PDQ} = 45^\circ$. Find the perimeter of ΔPBQ.

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5 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ (where $\mathbb{Z}$ is the set of all integers) such that

$$2000f(f(x)) - 3999f(x) + 1999x = 0 \text{ for all } x \in \mathbb{Z}.$$

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6 Let A_n be the number of ways to tile a $4 \times n$ rectangle using 2×1 tiles. Prove that A_n is divisible by 2 if and only if A_n is divisible by 3.
