South africa National Olympiad 2001

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by Arne

1 $ABCD$ is a convex quadrilateral with perimeter p. Prove that

$$\frac{1}{2}p < AC + BD < p.$$

(A polygon is convex if all of its interior angles are less than 180°.)

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2 Find all triples (x, y, z) of real numbers that satisfy

$$x\left(1-y^2\right)\left(1-z^2\right)+y\left(1-z^2\right)\left(1-x^2\right)+z\left(1-x^2\right)\left(1-y^2\right)$$
$$= 4xyz$$
$$= 4(x+y+z).$$

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3 For a certain real number x, the differences between x^{1919}, x^{1960} and x^{2001} are all integers. Prove that x is an integer.

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4 n red and n blue points on a plane are given so that no three of the $2n$ points are collinear. Prove that it is always possible to split up the points into n pairs, with one red and one blue point in each pair, so that no two of the n line segments which connect the two members of a pair intersect.

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5 Starting from a given cyclic quadrilateral $\mathcal{Q}_0$, a sequence of quadrilaterals is constructed so that $\mathcal{Q}_{k+1}$ is the circumscribed quadrilateral of $\mathcal{Q}_k$ for $k = 0, 1, \ldots$. The sequence terminates when a quadrilateral is reached that is not cyclic. (The circumscribed quadrilateral of a cylic quadrilateral $ABCD$ has sides that are tangent to the circumcircle of $ABCD$ at A, B, C and D.) Prove that the sequence always terminates, except when $\mathcal{Q}_0$ is a square.

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6 The unknown real numbers $x_1, x_2, \ldots, x_n$ satisfy $x_1 < x_2 < \cdots < x_n$, where $n \geq 3$. The numbers s, t and $d_1, d_2, \ldots, d_{n-2}$ are given, such that

$$s = \sum_{i=1}^{n} x_i,$$
$$t = \sum_{i=1}^{n} x_i^2,$$
$$d_i = x_{i+2} - x_i, \quad i = 1, 2, \ldots, n-2.$$

For which n is this information always sufficient to determine $x_1, x_2, \ldots, x_n$ uniquely?
