South africa National Olympiad 2002

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by Arne

1 Given a quadrilateral $ABCD$ such that $AB^2 + CD^2 = AD^2 + BC^2$, prove that $AC \perp BD$.

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2 Find all triples of natural numbers (a, b, c) such that a, b and c are in geometric progression and $a + b + c = 111$.

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3 A small square $PQRS$ is contained in a big square. Produce PQ to A, QR to B, RS to C and SP to D so that A, B, C and D lie on the four sides of the large square in order, produced if necessary. Prove that $AC = BD$ and $AC \perp BD$.

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4 How many ways are there to express 1000000 as a product of exactly three integers greater than 1? (For the purpose of this problem, abc is not considered different from bac, etc.)

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5 In acute-angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides. r_b and r_c are defined similarly. r is the radius of the incircle of ABC. Show that

$$\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

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6 Find all rational numbers a, b, c and d such that

$$8a^2 - 3b^2 + 5c^2 + 16d^2 - 10ab + 42cd + 18a + 22b - 2c - 54d = 42,$$

$$15a^2 - 3b^2 + 21c^2 - 5d^2 + 4ab + 32cd - 28a + 14b - 54c - 52d = -22.$$
