South africa National Olympiad 2003

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by Arne

1 You have five pieces of paper. You pick one or more of them and cut each of them into five smaller pieces. Now you take one or more of the pieces from this lot and cut each of these into five smaller pieces. And so on. Prove that you will never have 2003 pieces.

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2 Given a parallelogram $ABCD$, join A to the midpoints E and F of the opposite sides BC and CD. AE and AF intersect the diagonal BD in M and N. Prove that M and N divide BD into three equal parts.

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3 The first four digits of a certain positive integer n are 1137. Prove that the digits of n can be shuffled in such a way that the new number is divisible by 7.

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4 In a given pentagon $ABCDE$, triangles ABC, BCD, CDE, DEA and EAB all have the same area. The lines AC and AD intersect BE at points M and N. Prove that $BM = EN$.

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5 Prove that the sum of the squares of two consecutive positive integers cannot be equal to a sum of the fourth powers of two consecutive positive integers.

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6 In ΔABC, the sum of the sides is $2s$ and the radius of the incircle is r. Three semicircles with diameters AB, BC and CA are drawn on the outside of ABC. A circle with radius t touches all three semicircles. Prove that

$$\frac{s}{2} < t \leq \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$
