South africa National Olympiad 2004

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1 Let $a = 1111\ldots1111$ and $b = 1111\ldots1111$ where a has forty ones and b has twelve ones. Determine the greatest common divisor of a and b.

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2 Fifty points are chosen inside a convex polygon having eighty sides such that no three of the fifty points lie on the same straight line. The polygon is cut into triangles such that the vertices of the triangles are just the fifty points and the eighty vertices of the polygon. How many triangles are there?

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3 Find all real numbers x such that $x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor = 88$. The notation $\lfloor x\rfloor$ means the greatest integer less than or equal to x.

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4 Let A_1 and B_1 be two points on the base AB of isosceles triangle ABC (with $\widehat{C} > 60^\circ$) such that $\widehat{A_1CB_1} = \widehat{BAC}$. A circle externally tangent to the circumcircle of triangle $\triangle A_1B_1C$ is tangent also to rays CA and CB at points A_2 and B_2 respectively. Prove that $A_2B_2 = 2AB$.

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5 For $n \geq 2$, find the number of integers x with $0 \leq x < n$, such that x^2 leaves a remainder of 1 when divided by n.

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6 The numbers a_1, a_2 and a_3 are distinct positive integers, such that

(i) a_1 is a divisor of $a_2 + a_3 + a_2a_3$;

(ii) a_2 is a divisor of $a_3 + a_1 + a_3a_1$;

(iii) a_3 is a divisor of $a_1 + a_2 + a_1a_2$.

Prove that a_1, a_2 and a_3 cannot all be prime.
