South africa National Olympiad 2005

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1 Five numbers are chosen from the diagram below, such that no two numbers are chosen from the same row or from the same column. Prove that their sum is always the same.

1	4	7	10	13
16	19	22	25	28
31	34	37	40	43
46	49	52	55	58
61	64	67	70	73

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2 Let F be the set of all fractions m/n where m and n are positive integers with $m+n \leq 2005$. Find the largest number a in F such that $a < 16/23$.

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3 A warehouse contains 175 boots of size 8, 175 boots of size 9 and 200 boots of size 10. Of these 550 boots, 250 are for the left foot and 300 for the right foot. Let n denote the total number of usable pairs of boots in the warehouse. (A usable pair consists of a left and a right boot of the same size.)

(a) Is $n = 50$ possible?

(b) Is $n = 51$ possible?

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4 The inscribed circle of triangle ABC touches the sides BC, CA and AB at D, E and F respectively. Let Q denote the other point of intersection of AD and the inscribed circle. Prove that EQ extended passes through the midpoint of AF if and only if $AC = BC$.

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5 Let $x_1, x_2, \ldots, x_n$ be positive numbers with product equal to 1. Prove that there exists a $k \in \{1, 2, \ldots, n\}$ such that

$$\frac{x_k}{k + x_1 + x_2 + \cdots + x_k} \geq 1 - \frac{1}{\sqrt[n]{2}}.$$

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6 Consider the increasing sequence $1, 2, 4, 5, 7, 9, 10, 12, 14, 16, 17, 19, \ldots$ of positive integers, obtained by concatenating alternating blocks $\{1\}, \{2, 4\}, \{5, 7, 9\}, \{10, 12, 14, 16\}, \ldots$ of odd and even numbers. Each block contains one more element than the previous one and the first element in each block is one more than the last element of the previous one. Prove that the n-th element of the sequence is given by

$$2n - \left\lfloor \frac{1 + \sqrt{8n-7}}{2} \right\rfloor.$$

(Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)