South africa National Olympiad 2007

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by Raja Oktovin

1 Determine whether $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}$ is less than or greater than $\frac{3}{10}$.

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2 Consider the equation $x^4 = ax^3 + bx^2 + cx + 2007$, where a, b, c are real numbers. Determine the largest value of b for which this equation has exactly three distinct solutions, all of which are integers.

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3 In acute-angled triangle ABC, the points D, E, F are on sides BC, CA, AB, respectively such that $\angle AFE = \angle BFD, \angle FDB = \angle EDC, \angle DEC = \angle FEA$. Prove that AD is perpendicular to BC.

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4 Let ABC be a triangle and $PQRS$ a square with P on AB, Q on AC, and R and S on BC. Let H on BC such that AH is the altitude of the triangle from A to base BC. Prove that:

(a) $\frac{1}{AH} + \frac{1}{BC} = \frac{1}{PQ}$
(b) the area of ABC is twice the area of $PQRS$ iff $AH = BC$

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5 Let Z and R denote the sets of integers and real numbers, respectively.
Let $f : Z \to R$ be a function satisfying:
(i) $f(n) \geq 0$ for all $n \in Z$
(ii) $f(mn) = f(m)f(n)$ for all $m, n \in Z$
(iii) $f(m+n) \leq max(f(m), f(n))$ for all $m, n \in Z$

(a) Prove that $f(n) \leq 1$ for all $n \in Z$
(b) Find a function $f : Z \to R$ satisfying (i), (ii),(iii) and $0 < f(2) < 1$ and $f(2007) = 1$

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6 Prove that it is not possible to write numbers $1, 2, 3, ..., 25$ on the squares of 5x5 chessboard such that any neighboring numbers differ by at most 4.
