South africa National Olympiad 2008

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by djb86

1 Determine the number of positive divisors of 2008^8 that are less than 2008^4.

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2 Let $ABCD$ be a convex quadrilateral with the property that AB extended and CD extended intersect at a right angle. Prove that $AC \cdot BD > AD \cdot BC$.

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3 Let a, b, c be positive real numbers. Prove that

$$(a+b)(b+c)(c+a) \geq 8(a+b-c)(b+c-a)(c+a-b)$$

and determine when equality occurs.

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4 A pack of 2008 cards, numbered from 1 to 2008, is shuffled in order to play a game in which each move has two steps:

(i) the top card is placed at the bottom;

(ii) the new top card is removed.

It turns out that the cards are removed in the order $1, 2, \dots, 2008$. Which card was at the top before the game started?

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5 Triangle ABC has orthocentre H. The feet of the perpendiculars from H to the internal and external bisectors of $\hat{A}$ are P and Q respectively. Prove that P is on the line that passes through Q and the midpoint of BC. (Note: The ortohcentre of a triangle is the point where the three altitudes intersect.)

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6 Find all function pairs (f, g) where each f and g is a function defined on the integers and with values, such that, for all integers a and b,

$$f(a+b) = f(a)g(b) + g(a)f(b) g(a+b) = g(a)g(b) - f(a)f(b).$$
