South africa National Olympiad 2009

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by djb86

1 Determine the smallest integer $n > 1$ with the property that $n^2(n-1)$ is divisible by 2009.

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2 Let $ABCD$ be a rectangle and E the reflection of A with respect to the diagonal BD. If $EB = EC$, what is the ratio $\frac{AD}{AB}$?

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3 Ten girls, numbered from 1 to 10, sit at a round table, in a random order. Each girl then receives a new number, namely the sum of her own number and those of her two neighbours. Prove that some girl receives a new number greater than 17.

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4 Let $x_1, x_2, \dots, x_n$ be a finite sequence of real numbersm mwhere $0 < x_i < 1$ for all $i = 1, 2, \dots, n$. Put $P = x_1 x_2 \cdots x_n$, $S = x_1 + x_2 + \cdots + x_n$ and $T = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$. Prove that

$$\frac{T-S}{1-P} > 2.$$

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5 A game is played on a board with an infinite row of holes labelled $0, 1, 2, \dots$. Initially, 2009 pebbles are put into hole 1; the other holes are left empty. Now steps are performed according to the following scheme:

(i) At each step, two pebbles are removed from one of the holes (if possible), and one pebble is put into each of the neighbouring holes.

(ii) No pebbles are ever removed from hole 0.

(iii) The game ends if there is no hole with a positive label that contains at least two pebbles.

Show that the game always terminates, and that the number of pebbles in hole 0 at the end of the game is independent of the specific sequence of steps. Determine this number.

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6 Let A denote the set of real numbers x such that $0 \le x < 1$. A function $f : A \to \mathbb{R}$ has the properties:

(i) $f(x) = 2f(\frac{x}{2})$ for all $x \in A$;

(ii) $f(x) = 1 - f(x - \frac{1}{2})$ if $\frac{1}{2} \le x < 1$.

Prove that

(a) $f(x) + f(1-x) \ge \frac{2}{3}$ if x is rational and $0 < x < 1$.

(b) There are infinitely many odd positive integers q such that equality holds in (a) when $x = \frac{1}{q}$.