South africa National Olympiad 2012

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by DylanN

1 Given that $\frac{1+3+5+\cdots+(2n-1)}{2+4+6+\cdots+(2n)}=\frac{2011}{2012}$,
determine n.

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2 Let $ABCD$ be a square and X a point such that A and X are on opposite sides of CD. The lines AX and BX intersect CD in Y and Z respectively. If the area of $ABCD$ is 1 and the area of XYZ is $\frac{2}{3}$, determine the length of YZ

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3 Sixty points, of which thirty are coloured red, twenty are coloured blue and ten are coloured green, are marked on a circle. These points divide the circle into sixty arcs. Each of these arcs is assigned a number according to the colours of its endpoints: an arc between a red and a green point is assigned a number 1, an arc between a red and a blue point is assigned a number 2, and an arc between a blue and a green point is assigned a number 3. The arcs between two points of the same colour are assigned a number 0. What is the greatest possible sum of all the numbers assigned to the arcs?

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4 Let p and k be positive integers such that p is prime and $k>1$. Prove that there is at most one pair (x,y) of positive integers such that $x^k+px=y^k$.

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5 Let ABC be a triangle such that $AB\neq AC$. We denote its orthocentre by H, its circumcentre by O and the midpoint of BC by D. The extensions of HD and AO meet in P. Prove that triangles AHP and ABC have the same centroid.

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6 Find all functions $f:\mathbb{N}\to\mathbb{R}$ such that $f(km)+f(kn)-f(k)f(mn)\geq 1$
for all $k,m,n\in\mathbb{N}$.
