IMTS 1991

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Day 1

1 For every positive integer n, form the number $n/s(n)$, where $s(n)$ is the sum of digits of n in base 10. Determine the minimum value of $n/s(n)$ in each of the following cases:

(i) $10 \leq n \leq 99$
(ii) $100 \leq n \leq 999$
(iii) $1000 \leq n \leq 9999$
(iv) $10000 \leq n \leq 99999$

2 Find all pairs of integers, n and k, $2 < k < n$, such that the binomial coefficients

$$\binom{n}{k-1}, \binom{n}{k}, \binom{n}{k+1}$$

form an increasing arithmetic series.

3 On a 8 x 8 board we place n dominoes, each covering two adjacent squares, so that no more dominoes can be placed on the remaining squares. What is the smallest value of n for which the above statement is true?

4 Show that an arbitary triangle can be dissected by straight line segments into three parts in three different ways so that each part has a line of symmetry.

5 Show that it is impossible to dissect an arbitary tetrahedron into six parts by planes or portions thereof so that each of the parts has a plane of symmetry.

Day 2

1 What is the smallest integer multiple of 9997, other than 9997 itself, which contains only odd digits?

2 Show that every triangle can be dissected into nine convex nondegenrate pentagons.

3 Prove that if x, y and z are pairwise relatively prime positive integers, and if $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$, then $x + y, x - z, y - z$ are perfect squares of integers.

4 Let a, b, c, d be the areas of the triangular faces of a tetrahedron, and let h_a, h_b, h_c, h_d be the corresponding altitudes of the tetrahedron. If V denotes the volume of tetrahedron, prove that

$$(a+b+c+d)(h_a+h_b+h_c+h_d) \geq 48V$$

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5 Prove that there are infinitely many positive integers n such that $n \times n \times n$ can not be filled completely with 2 x 2 x 2 and 3 x 3 x 3 solid cubes.

Day 3

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1 Note that if the product of any two distinct members of 1,16,27 is increased by 9, the result is the perfect square of an integer. Find the unique positive integer n for which $n+9, 16n+9, 27n+9$ are also perfect squares.

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2 Note that 1990 can be "turned into a square" by adding a digit on its right, and some digits on its left; i.e., $419904 = 648^2$. Prove that 1991 cannot be turned into a square by the same procedure; i.e., there are no digits $d, x, y, ..$ such that $...yx1991d$ is a perfect square.

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3 Find k if $P, Q, R,$ and S are points on the sides of quadrilateral $ABCD$ so that

$$\frac{AP}{PB} = \frac{BQ}{QC} = \frac{CR}{RD} = \frac{DS}{SA} = k,$$

and the area of the quadrilateral $PQRS$ is exactly 52

For picture, go here (`http://www.cms.math.ca/Competitions/IMTS/imts3.html`).

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4 Let n points with integer coordinates be given in the xy-plane. What is the minimum value of n which will ensure that three of the points are the vertices of a triangel with integer (possibly, 0) area?

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5 Two people, A and B, play the following game with a deck of 32 cards. With A starting, and thereafter the players alternating, each player takes either 1 card or a prime number of cards. Eventually all of the cards are chosen, and the person who has none to pick up is the loser. Who will win the game if they both follow optimal strategy?

Day 4

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1 Use each of the digits 1,2,3,4,5,6,7,8,9 exactly twice to form distinct prime numbers whose sum is as small as possible. What must this minimal sum be? (Note: The five smallest primes are 2,3,5,7, and 11)

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2 Find the smallest positive integer, n, which can be expressed as the sum of distinct positive integers a, b, c such that $a+b, a+c, b+c$ are perfect squares.

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3 Prove that a positive integer can be expressed in the form $3x^2 + y^2$ iff it can also be expressed in form $u^2 + uv + v^2$, where x, y, u, v are all positive integers.

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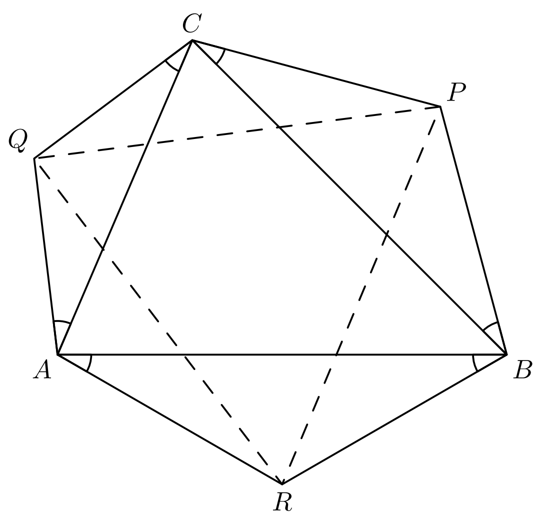
4 Let $\triangle ABC$ be an arbitary triangle, and construct P, Q, R so that each of the angles marked is 30°. Prove that $\triangle PQR$ is an equilateral triangle.



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5 The sides of $\triangle ABC$ measure 11,20, and 21 units. We fold it along PQ, QR, RP where P, Q, R are the midpoints of its sides until A, B, C coincide. What is the volume of the resulting tetrahedron?
