IMTS 1992

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Day 1

1 The set S consists of five integers. If pairs of distinct elements of S are added, the following ten sums are obtained: 1967,1972,1973,1974,1975,1980,1983,1984,1989,1991. What are the elements of S?

2 Let $n \geq 3$ and $k \geq 2$ be integers, and form the forward differences of the members of the sequence $1, n, n^2, ...n^{k-1}$

and successive forward differences thereof, as illustrated on the right for case $(n,k) = (3,5)$. Prove that all entries of the resulting triangles of positive integers are distinct from one another.

Diagram:
http://www.cms.math.ca/Competitions/IMTS/imts5.html

3 In a mathematical version of baseball, the umpire chooses a positive integer m, $m \leq n$, and you guess positive integers to obtain information about m. If your guess is smaller than the umpire's m, he calls it a "ball"; if it is greater than or equal to m, he calls it a "strike." To "hit" it you must state the the correct value of m after the 3rd strike or the 6th guess, whichever comes first. What is the largest n so that there exists a strategy that will allow you to bat 1.000, i.e. always state m correctly? Describe your strategy in detail.

4 Prove that if f is a non-constant real-valued function such that for all real x, $f(x+1)+f(x-1) = \sqrt{3}f(x)$, then f is periodic. What is the smallest p, $p > 0$ such that $f(x+p) = f(x)$ for all x?

5 In $\triangle ABC$, shown on the right, let r denote the radius of the inscribed circle, and let r_A, r_B, and r_C denote the radii of the smaller circles tangent to the inscribed circle and to the sides emanating from A, B, and C, respectively. Prove that

$r \leq r_A + r_B + r_C$

Day 2

1 Nine lines, parallel to the base of a triangle, divide the other sides into 10 equal segments and the area into 10 distinct parts. Find the area of the original triangle, if the area of the largest of these parts is 76.

2 In how many ways can 1992 be expressed as the sum of one or more consecutive integers?

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3 Show that there exists an equiangular hexagon in the plane, whose sides measure 5,8,11,14,23, and 29 units in some order.

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4 An international firm has 250 employees, each of whom speaks several languages. For each pair of employees, (A, B), there is a language spoken by A and not B, and there is another language spoken by B but not A. At least how many languages must be spoken at the firm?

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5 An infinite checkerboard is divided by a horizontal line into upper and lower halves as shown on the right. A number of checkers are to be placed on the board below the line (within the squares). A "move" consists of one checker jumping horizontally or vertically over a second checker, and removing the second checker. What is the minimum value of n which will allow the placement of the last checker in row 4 above the dividing horizontal line after $n-1$ moves? Describe the initial position of the checkers as well as each of the moves.

Picture: http://www.cms.math.ca/Competitions/IMTS/imts6.gif

Day 3

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1 In trapezoid $ABCD$, the diagonals intersect at E, the area of $\triangle ABE$ is 72 and the area of $\triangle CDE$ is 50. What is the area of trapezoid $ABCD$?

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2 Prove that if a, b, c are positive integers such that $c^2 = a^2 + b^2$, then both $c^2 + ab$ and $c^2 - ab$ are also expressible as the sums of squares of two positive integers.

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3 For n a positive integer, denote by $P(n)$ the product of all positive integers divisors of n. Find the smallest n for which

$$P(P(P(n))) > 10^{12}$$

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4 In an attempt to copy down from the board a sequence of six positive integers in arithmetic progression, a student wrote down the five numbers,

$$113, 137, 149, 155, 173,$$

accidentally omitting one. He later discovered that he also miscopied one of them. Can you help him and recover the original sequence?

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5 Let $T = (a, b, c)$ be a triangle with sides a, b and c and area $\triangle$. Denote by $T' = (a', b', c')$ the triangle whose sides are the altitudes of T (i.e., $a' = h_a, b' = h_b, c' = h_c$) and denote its area by $\triangle'$. Similarly, let $T'' = (a'', b'', c'')$ be the triangle formed from the altitudes of T', and denote its area by $\triangle''$. Given that $\triangle' = 30$ and $\triangle'' = 20$, find $\triangle$.