USA Team Selection Test 2000

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Day 1 June 10th

1 Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}.$$

2 Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.

3 Let p be a prime number. For integers r, s such that $rs(r^2-s^2)$ is not divisible by p, let $f(r, s)$ denote the number of integers $n \in \{1, 2, \ldots, p-1\}$ such that $\{rn/p\}$ and $\{sn/p\}$ are either both less than $1/2$ or both greater than $1/2$. Prove that there exists $N > 0$ such that for $p \geq N$ and all r, s,

$$\left\lceil \frac{p-1}{3} \right\rceil \leq f(r, s) \leq \left\lfloor \frac{2(p-1)}{3} \right\rfloor.$$

Day 2 June 11th

4 Let n be a positive integer. Prove that

$$\binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1} = \frac{n+1}{2^{n+1}} \left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1} \right).$$

5 Let n be a positive integer. A $corner$ is a finite set S of ordered n-tuples of positive integers such that if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are positive integers with $a_k \geq b_k$ for $k = 1, 2, \ldots, n$ and $(a_1, a_2, \ldots, a_n) \in S$, then $(b_1, b_2, \ldots, b_n) \in S$. Prove that among any infinite collection of corners, there exist two corners, one of which is a subset of the other one.

6 Let ABC be a triangle inscribed in a circle of radius R, and let P be a point in the interior of triangle ABC. Prove that

$$\frac{PA}{BC^2} + \frac{PB}{CA^2} + \frac{PC}{AB^2} \geq \frac{1}{R}.$$

Alternative formulation: If ABC is a triangle with sidelengths $BC = a$, $CA = b$, $AB = c$ and circumradius R, and P is a point inside the triangle ABC, then prove that
$\frac{PA}{a^2} + \frac{PB}{b^2} + \frac{PC}{c^2} \geq \frac{1}{R}$.