USA Team Selection Test 2001

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Day 1 June 9th

1 Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers such that $a_{n+1}\geq a_n^2+\frac{1}{5}$ for all $n\geq 0$. Prove that $\sqrt{a_{n+5}}\geq a_{n-5}^2$ for all $n\geq 5$.

2 Express

$$\sum_{k=0}^{n}(-1)^k(n-k)!(n+k)!$$

in closed form.

3 For a set S, let $|S|$ denote the number of elements in S. Let A be a set of positive integers with $|A|=2001$. Prove that there exists a set B such that

(i) $B\subseteq A$;
(ii) $|B|\geq 668$;
(iii) for any $u,v\in B$ (not necessarily distinct), $u+v\notin B$.

Day 2 June 10th

4 There are 51 senators in a senate. The senate needs to be divided into n committees so that each senator is on one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does *not* necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

5 In triangle ABC, $\angle B=2\angle C$. Let P and Q be points on the perpendicular bisector of segment BC such that rays AP and AQ trisect $\angle A$. Prove that $PQ<AB$ if and only if $\angle B$ is obtuse.

6 Let a,b,c be positive real numbers such that

$$a+b+c\geq abc.$$

Prove that at least two of the inequalities

$$\frac{2}{a}+\frac{3}{b}+\frac{6}{c}\geq 6,\qquad \frac{2}{b}+\frac{3}{c}+\frac{6}{a}\geq 6,\qquad \frac{2}{c}+\frac{3}{a}+\frac{6}{b}\geq 6$$

are true.

Day 3

7 Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle ADC = 135^\circ$ and

$$AC^2 \cdot BD^2 = 2 \cdot AB \cdot BC \cdot CD \cdot DA.$$

Prove that the diagonals of the quadrilateral $ABCD$ are perpendicular.

8 Find all pairs of nonnegative integers (m, n) such that

$$(m+n-5)^2 = 9mn.$$

9 Let A be a finite set of positive integers. Prove that there exists a finite set B of positive integers such that $A \subseteq B$ and

$$\prod_{x \in B} x = \sum_{x \in B} x^2.$$

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