Spain Mathematical Olympiad 2001

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– Session 1

Problem 1 Prove that the graph of the polynomial $P(x)$ is symmetric in respect to point $A(a, b)$ if and only if there exists a polynomial $Q(x)$ such that: $P(x) = b + (x - a)Q((x - a)^2))$.

Problem 2 Let P be a point on the interior of triangle ABC, such that the triangle ABP satisfies $AP = BP$. On each of the other sides of ABC, build triangles BQC and CRA exteriorly, both similar to triangle ABP satisfying:

$$BQ = QC$$

and

$$CR = RA.$$

Prove that the point $P, Q, C,$ and R are collinear or are the vertices of a parallelogram.

Problem 3 You have five segments of lengths $a_1, a_2, a_3, a_4,$ and a_5 such that it is possible to form a triangle with any three of them. Demonstrate that at least one of those triangles has angles that are all acute.

– Session 2

Problem 4 The integers between 1 and 9 inclusive are distributed in the units of a 3 x 3 table. You sum six numbers of three digits: three that are read in the rows from left to right, and three that are read in the columns from top to bottom. Is there any such distribution for which the value of this sum is equal to 2001?

Problem 5 A quadrilateral $ABCD$ is inscribed in a circle of radius 1 whose diameter is AB. If the quadrilateral $ABCD$ has an incircle, prove that $CD \leq 2\sqrt{5} - 2$.

Problem 6 Define the function $f : \mathbb{N} \to \mathbb{N}$ which satisfies, for any $s, n \in \mathbb{N}$, the following conditions: $f(1) = f(2^s)$ and if $n < 2^s$, then $f(2^s + n) = f(n) + 1$.
Calculate the maximum value of $f(n)$ when $n \leq 2001$ and find the smallest natural number n such that $f(n) = 2001$.