USA Team Selection Test 2003

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Day 1 June 20th

1 For a pair of integers a and b, with $0 < a < b < 1000$, set $S \subseteq \{1, 2, \ldots, 2003\}$ is called a *skipping set* for (a, b) if for any pair of elements $s_1, s_2 \in S$, $|s_1 - s_2| \notin \{a, b\}$. Let $f(a, b)$ be the maximum size of a skipping set for (a, b). Determine the maximum and minimum values of f.

2 Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, PC intersect sides BC, CA, AB at D, E, F, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$

if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)

3 Find all ordered triples of primes (p, q, r) such that

$$p \mid q^r + 1, \quad q \mid r^p + 1, \quad r \mid p^q + 1.$$

Reid Barton

Day 2 June 21st

4 Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(m+n)f(m-n) = f(m^2)$$

for $m, n \in \mathbb{N}$.

5 Let A, B, C be real numbers in the interval $\left(0, \frac{\pi}{2}\right)$. Let

$$X = \frac{\sin A \sin(A-B)\sin(A-C)}{\sin(B+C)}$$
$$Y = \frac{\sin B \sin(B-C)\sin(B-A)}{\sin(C+A)}$$
$$Z = \frac{\sin C \sin(C-A)\sin(C-B)}{\sin(A+B)}.$$

Prove that $X + Y + Z \geq 0$.

6 Let $\overline{AH_1}, \overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute scalene triangle ABC. The incircle of triangle ABC is tangent to $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at T_1, T_2, and T_3, respectively. For $k = 1, 2, 3$, let P_i be the point on line H_iH_{i+1} (where $H_4 = H_1$) such that $H_iT_iP_i$ is an acute isosceles triangle with $H_iT_i = H_iP_i$. Prove that the circumcircles of triangles $T_1P_1T_2$, $T_2P_2T_3$, $T_3P_3T_1$ pass through a common point.