USA Team Selection Test 2004

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Day 1

1 Suppose $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are real numbers such that

$$(a_1^2 + a_2^2 + \cdots + a_n^2 - 1)(b_1^2 + b_2^2 + \cdots + b_n^2 - 1) > (a_1b_1 + a_2b_2 + \cdots + a_nb_n - 1)^2.$$

Prove that $a_1^2 + a_2^2 + \cdots + a_n^2 > 1$ and $b_1^2 + b_2^2 + \cdots + b_n^2 > 1$.

2 Assume n is a positive integer. Considers sequences $a_0, a_1, \dots, a_n$ for which $a_i \in \{1, 2, \dots, n\}$ for all i and $a_n = a_0$.

(a) Suppose n is odd. Find the number of such sequences if $a_i - a_{i-1} \not\equiv i \pmod{n}$ for all $i = 1, 2, \dots, n$.

(b) Suppose n is an odd prime. Find the number of such sequences if $a_i - a_{i-1} \not\equiv i, 2i \pmod{n}$ for all $i = 1, 2, \dots, n$.

3 Draw a 2004×2004 array of points. What is the largest integer n for which it is possible to draw a convex n-gon whose vertices are chosen from the points in the array?

Day 2

4 Let ABC be a triangle. Choose a point D in its interior. Let ω_1 be a circle passing through B and D and ω_2 be a circle passing through C and D so that the other point of intersection of the two circles lies on AD. Let ω_1 and ω_2 intersect side BC at E and F, respectively. Denote by X the intersection of DF, AB and Y the intersection of DE, AC. Show that $XY \parallel BC$.

5 Let $A = (0, 0, 0)$ in 3D space. Define the *weight* of a point as the sum of the absolute values of the coordinates. Call a point a *primitive lattice point* if all of its coordinates are integers whose gcd is 1. Let square $ABCD$ be an *unbalanced primitive integer square* if it has integer side length and also, B and D are primitive lattice points with different weights. Prove that there are infinitely many unbalanced primitive integer squares such that the planes containing the squares are not parallel to each other.

6 Define the function $f : \mathbb{N} \cup \{0\} \to \mathbb{Q}$ as follows: $f(0) = 0$ and

$$f(3n + k) = -\frac{3f(n)}{2} + k,$$

for $k = 0, 1, 2$. Show that f is one-to-one and determine the range of f.

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