USA Team Selection Test 2014

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–	December TST
1	Let ABC be an acute triangle, and let X be a variable interior point on the minor arc BC of its circumcircle. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point. (Specifically: prove that there is a point F, determined by triangle ABC, such that no matter where X is on arc BC, line ℓ passes through F.) *Robert Simson et al.*
2	Let $a_1, a_2, a_3, \ldots$ be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for every positive integers n and k, the quantity $$\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}$$ is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to the same perfect square). *Evan O'Dorney and Victor Wang*
3	Let n be an even positive integer, and let G be an n-vertex graph with exactly $\frac{n^2}{4}$ edges, where there are no loops or multiple edges (each unordered pair of distinct vertices is joined by either 0 or 1 edge). An unordered pair of distinct vertices $\{x, y\}$ is said to be *amicable* if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ pairs of vertices which are amicable. *Zoltán Füredi (suggested by Po-Shen Loh)*
–	January TST
1	Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying $$\sum_{i=1}^{n-1} \lvert c_i - 1 \rvert < 1.$$ Prove that $$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$ has no real roots.

2 Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, and DA respectively. Let W, X, Y and Z be the orthocenters of triangles AHE, BEF, CFG and DGH, respectively. Prove that the quadrilaterals $ABCD$ and $WXYZ$ have the same area.

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3 For a prime p, a subset S of residues modulo p is called a *sum-free multiplicative subgroup* of $\mathbb{F}_p$ if • there is a nonzero residue α modulo p such that $S = \{1, \alpha^1, \alpha^2, \dots\}$ (all considered mod p), and • there are no $a, b, c \in S$ (not necessarily distinct) such that $a + b \equiv c \pmod p$.
Prove that for every integer N, there is a prime p and a sum-free multiplicative subgroup S of $\mathbb{F}_p$ such that $|S| \ge N$.

Proposed by Noga Alon and Jean Bourgain
