USAMTS Problems 1998

www.artofproblemsolving.com/community/c4672

by JSGandora

– Round 1

1 Several pairs of positive integers (m, n) satisfy the condition $19m + 90 + 8n = 1998$. Of these, $(100, 1)$ is the pair with the smallest value for n. Find the pair with the smallest value for m.

2 Determine the smallest rational number $\frac{r}{s}$ such that $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} \le \frac{r}{s}$ whenever $k, m,$ and n are positive integers that satisfy the inequality $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$.

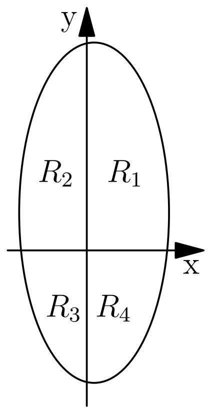
3 It is possible to arrange eight of the nine numbers $2, 3, 4, 7, 10, 11, 12, 13, 15$ in the vacant squares of the 3 by 4 array shown on the right so that the arithmetic average of the numbers in each row and in each column is the same integer. Exhibit such an arrangement, and specify which one of the nine numbers must be left out when completing the array.

1			
	9		5
		14	

4 Show that it is possible to arrange seven distinct points in the plane so that among any three of these seven points, two of the points are a unit distance apart. (Your solution should include a carefully prepared sketch of the seven points, along with all segments that are of unit length.)

5 The gure on the right shows the ellipse $\frac{(x-19)^2}{19} + \frac{(x-98)^2}{98} = 1998.$
Let $R_1, R_2, R_3,$ and R_4 denote those areas within the ellipse that are in the rst, second, third, and fourth quadrants, respectively. Determine the value of $R_1 - R_2 + R_3 - R_4$.

© 2019 AoPS Incorporated



– Round 2

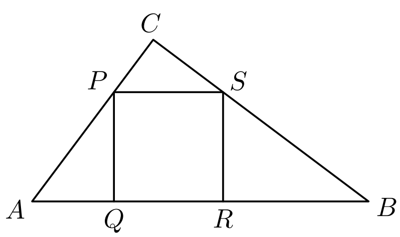
1 Determine the unique pair of real numbers (x, y) that satisfy the equation

$$(4x^2 + 6x + 4)(4y^2 - 12y + 25) = 28.$$

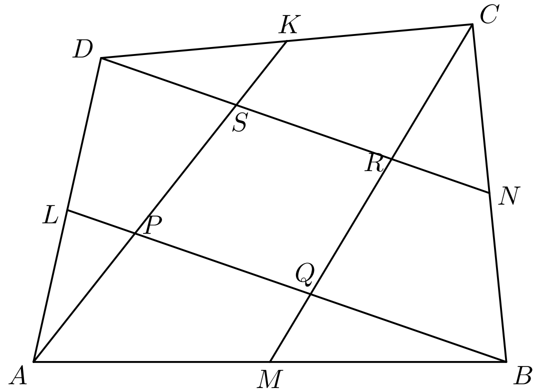
2 Prove that there are innitely many ordered triples of positive integers (a, b, c) such that the greatest common divisor of $a, b,$ and c is 1, and the sum $a^2b^2 + b^2c^2 + c^2a^2$ is the square of an integer.

3 Nine cards can be numbered using positive half-integers $(1/2, 1, 3/2, 2, 5/2, \dots)$ so that the sum of the numbers on a randomly chosen pair of cards gives an integer from 2 to 12 with the same frequency of occurrence as rolling that sum on two standard dice. What are the numbers on the nine cards and how often does each number appear on the cards?

4 As shown on the gure, square $PQRS$ is inscribed in right triangle ABC, whose right angle is at C, so that S and P are on sides BC and CA, respectively, while Q and R are on side AB. Prove that $AB \geq 3QR$ and determine when equality occurs.



5 In the gure on the right, $ABCD$ is a convex quadrilateral, $K, L, M,$ and N are the midpoints of its sides, and $PQRS$ is the quadrilateral formed by the intersections of $AK, BL, CM,$ and DN. Determine the area of quadrilateral $PQRS$ if the area of quadrilateral $ABCD$ is 3000, and the areas of quadrilaterals $AMQP$ and $CKSR$ are 513 and 388, respectively.



– Round 3

---

1 Determine the leftmost three digits of the number

$$1^1 + 2^2 + 3^3 + ... + 999^{999} + 1000^{1000}.$$

---

2 There are innitely many ordered pairs (m, n) of positive integers for which the sum

$$m + (m + 1) + (m + 2) + ... + (n - 1) + n$$

is equal to the product mn. The four pairs with the smallest values of m are $(1, 1), (3, 6), (15, 35),$ and $(85, 204)$. Find three more (m, n) pairs.

---

3 The integers from 1 to 9 can be arranged into a 3×3 array (as shown on the right) so that the sum of the numbers in every row, column, and diagonal is a multiple of 9.

(a.) Prove that the number in the center of the array must be a multiple of 3.

(b.) Give an example of such an array with 6 in the center.

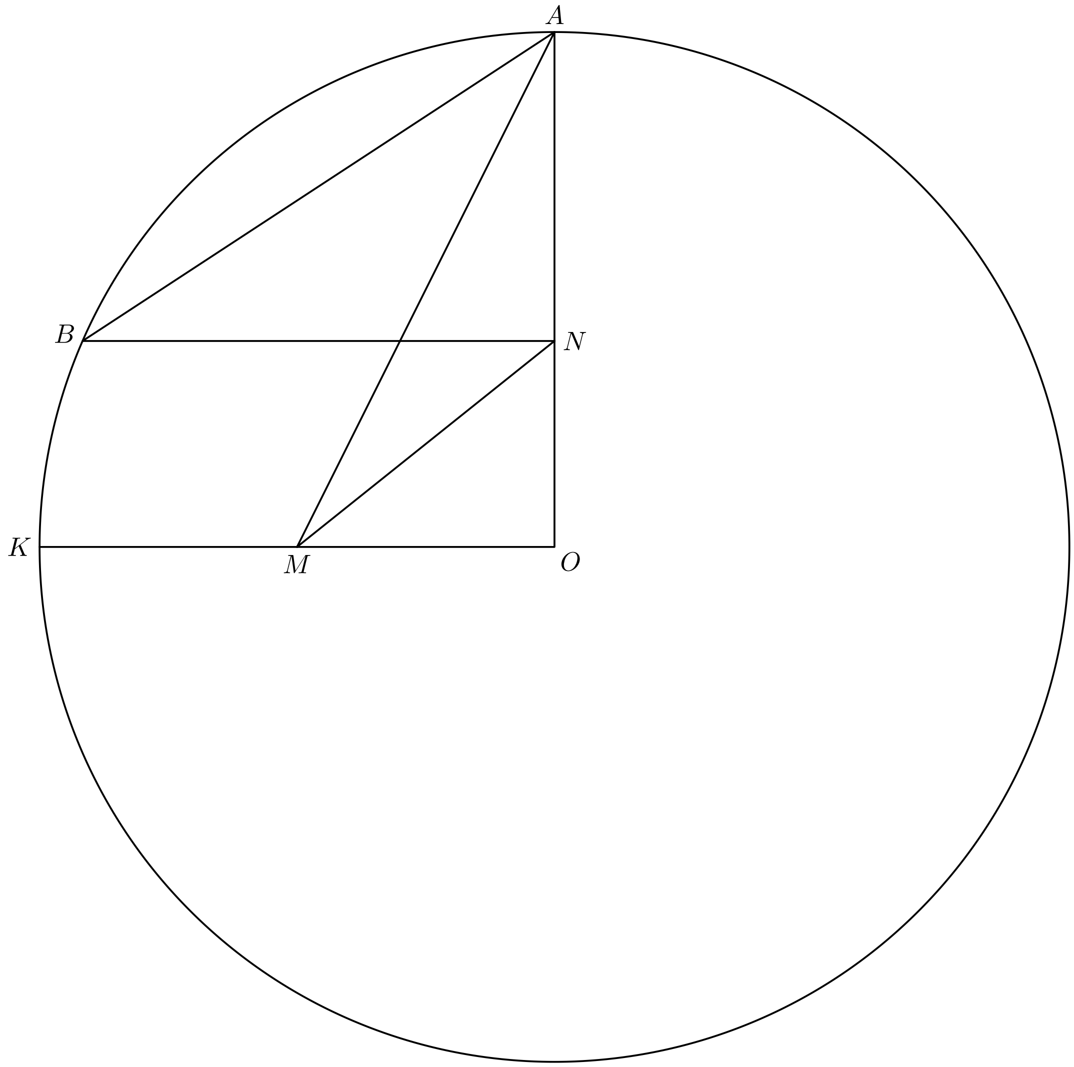
A	B	C
D	E	F
G	H	I

---

4 Prove that if $0 < x < \pi/2$, then $\sec^6 x + \csc^6 x + (\sec^6 x)(\csc^6 x) \geq 80$.

---

5 In the gure on the right, O is the center of the circle, OK and OA are perpendicular to one another, M is the midpoint of OK, BN is parallel to OK, and $\angle AMN = \angle NMO$. Determine the measure of $\angle ABN$ in degrees.



– Round 4

1 Exhibit a 13-digit integer N that is an integer multiple of 2^{13} and whose digits consist of only 8s and 9s.

2 For a nonzero integer i, the exponent of 2 in the prime factorization of i is called $ord_2(i)$. For example, $ord_2(9) = 0$ since 9 is odd, and $ord_2(28) = 2$ since $28 = 2^2 \times 7$. The numbers $3^n - 1$ for $n = 1, 2, 3, \ldots$ are all even so $ord_2(3^n - 1) > 0$ for $n > 0$.
a) For which positive integers n is $ord_2(3^n - 1) = 1$?
b) For which positive integers n is $ord_2(3^n - 1) = 2$?
c) For which positive integers n is $ord_2(3^n - 1) = 3$?

Prove your answers.

---

3 Let f be a polynomial of degree 98, such that $f(k) = \frac{1}{k}$ for $k = 1, 2, 3, \ldots, 99$. Determine $f(100)$.

---

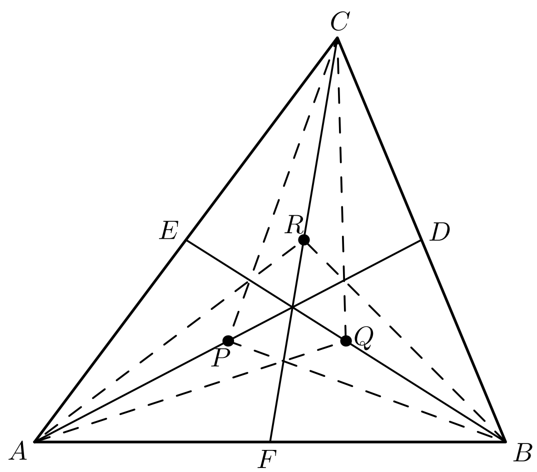
4 Let A consist of 16 elements of the set $\{1, 2, 3, \ldots, 106\}$, so that no two elements of A differ by $6, 9, 12, 15, 18,$ or 21. Prove that two elements of A must differ by 3.

---

5 In $\triangle ABC$, let $D, E,$ and F be the midpoints of the sides of the triangle, and let $P, Q,$ and R be the midpoints of the corresponding medians, $AD, BE,$ and CF, respectively, as shown in the gure at the right. Prove that the value of

$$\frac{AQ^2 + AR^2 + BP^2 + BR^2 + CP^2 + CQ^2}{AB^2 + BC^2 + CA^2}$$

does not depend on the shape of $\triangle ABC$ and nd that value.



© 2019 AoPS Incorporated