USAMTS Problems 1999

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by Binomial-theorem, djmathman

– Round 1

1 The digits of the three-digit integers $a, b,$ and c are the nine nonzero digits $1, 2, 3, \cdots 9$ each of them appearing exactly once. Given that the ratio $a : b : c$ is $1 : 3 : 5$, determine $a, b,$ and c.

2 Let $N = 111...1222...2$, where there are 1999 digits of 1 followed by 1999 digits of 2. Express N as the product of four integers, each of them greater than 1.

3 Triangle ABC has angle A measuring 30°, angle B measuring 60°, and angle C measuring 90°. Show four different ways to divide triangle ABC into four triangles, each similar to triangle ABC, but with one quarter of the area. Prove that the angles and sizes of the smaller triangles are correct.

4 There are 8436 steel balls, each with radius 1 centimeter, stacked in a tetrahedral pile, with one ball on top, 3 balls in the second layer, 6 in the third layer, 10 in the fourth, and so on. Determine the height of the pile in centimeters.

5 In a convex pentagon $ABCDE$ the sides have lengths $1, 2, 3, 4,$ and 5, though not necessarily in that order. Let $F, G, H,$ and I be the midpoints of the sides AB, BC, CD, and DE, respectively. Let X be the midpoint of segment FH, and Y be the midpoint of segment GI. The length of segment XY is an integer. Find all possible values for the length of side AE.

– Round 2

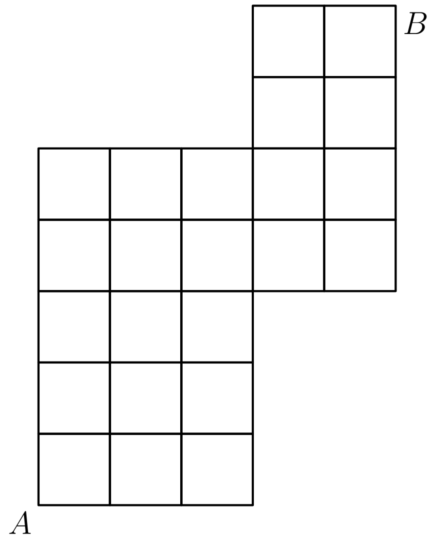
1 The number N consists of 1999 digits such that if each pair of consecutive digits in N were viewed as a two-digit number, then that number would either be a multiple of 17 or a multiple of 23. THe sum of the digits of N is 9599. Determine the rightmost ten digits of N.

2 Let C be the set of non-negative integers which can be expressed as $1999s + 2000t$, where s and t are also non-negative integers.

(a) Show that $3,994,001$ is not in C.

(b) Show that if $0 \le n \le 3,994,001$ and n is an integer not in C, then $3,994,001 - n$ is in C.

3 The figure on the right shows the map of Squareville, where each city block is of the same length. Two friends, Alexandra and Brianna, live at the corners marked by A and B, respectively. They start walking toward each other's house, leaving at the same time, walking with the same speed, and independently choosing a path to the other's house with uniform distribution out of all possible minimum-distance paths [that is, all minimum-distance paths are equally likely]. What is the probability they will meet?



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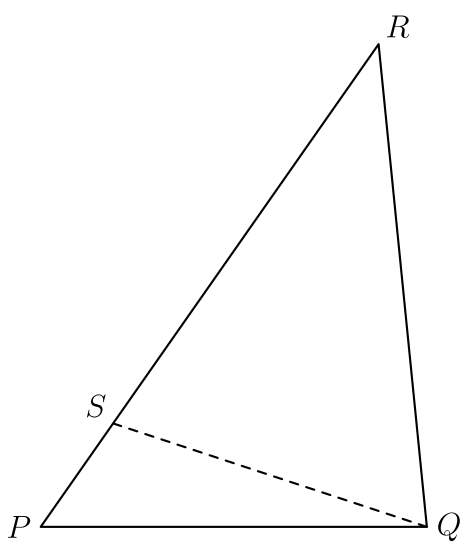
4 In $\triangle PQR$, $PQ = 8$, $QR = 13$, and $RP = 15$. Prove that there is a point S on line segment $\overline{PR}$, but not at its endpoints, such that PS and QS are also integers.



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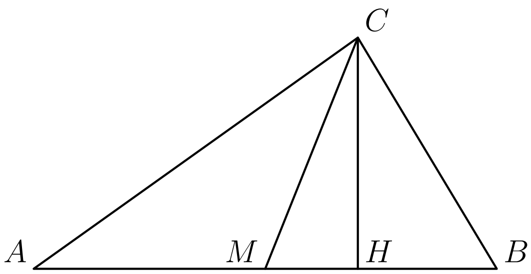
5 In $\triangle ABC$, $AC > BC$, CM is the median, and CH is the altitude emanating from C, as shown in the figure on the right. Determine the measure of $\angle MCH$ if $\angle ACM$ and $\angle BCH$ each have measure $17°$.



– Round 3

1 We define the *repetition* number of a positive integer n to be the number of distinct digits of n when written in base 10. Prove that each positive integer has a multiple which has a repetition number less than or equal to 2.

2 Let a be a positive real number, n a positive integer, and define the *power tower* $a \uparrow n$ recursively with $a \uparrow 1 = a$, and $a \uparrow (i+1) = a^{a\uparrow i}$ for $i = 1, 2, 3, \ldots$. For example, we have $4 \uparrow 3 = 4^{(4^4)} = 4^{256}$, a number which has 155 digits. For each positive integer k, let x_k denote the unique positive real number solution of the equation $x \uparrow k = 10 \uparrow (k+1)$. Which is larger: x_{42} or x_{43}?

3 Suppose that the 32 computers in a certain network are numbered with the 5-bit integers $00000, 00001, 00010, ..., 11111$ (bit is short for binary digit). Suppose that there is a one-way connection from computer A to computer B if and only if A and B share four of their bits with the remaining bit being 0 at A and 1 at B. (For example, 10101 can send messages to 11101 and to 10111.) We say that a computer is at level k in the network if it has exactly k 1s in its label $(k = 0, 1, 2, ..., 5)$. Suppose further that we know that 12 computers, three at each of the levels 1, 2, 3, and 4, are malfunctioning, but we do not know which ones. Can we still be sure that we can send a message from 00000 to 11111?

4 We say a triangle in the coordinate plane is *integral* if its three vertices have integer coordinates and if its three sides have integer lengths.

(a) Find an integral triangle with perimeter of 42.

(b) Is there an integral triangle with perimeter of 43?

5 We say that a finite set of points is *well scattered* on the surface of a sphere if every open hemisphere (half the surface of the sphere without its boundary) contains at least one of the points. The set $\{(1,0,0), (0,1,0), (0,0,1)\}$ is not well scattered on the unit sphere (the sphere of radius 1 centered at the origin), but if you add the correct point P it becomes well scattered. Find, with proof, all possible points P that would make the set well scattered.

– Round 4

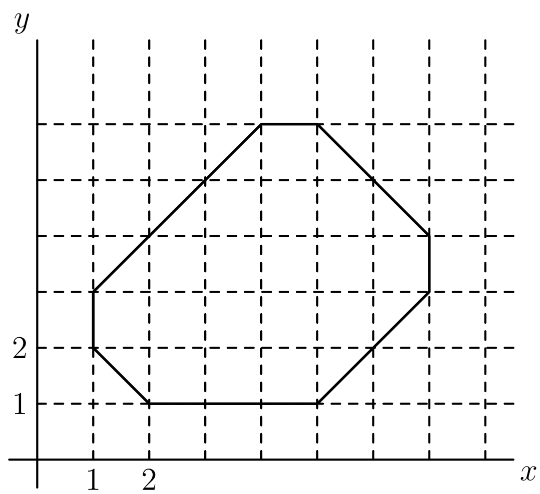
1 Define the unique 9-digit integer M that has the following properties: (1) its digits are all distinct and nonzero; and (2) for every positive integer $m = 2, 3, 4, ..., 9$, the integer formed by the leftmost m digits of M is divisible by m.

2 The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. It is well-known that the sum of any 10 consecutive Fibonacci numbers is divisible by 11. Determine the smallest integer N so that the sum of any N consecutive Fibonacci numbers is divisible by 12.

3 Determine the value of

$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}$$

4 We will say that an octagon is integral if its is equiangular, its vertices are lattice points (i.e., points with integer coordinates), and its area is an integer. For example, the figure on the right shows an integral octagon of area 21. Determine, with proof, the smallest positive integer K so that for every positive integer $k \geq K$, there is an integral octagon of area k.



5 (Revised 2-4-2000) Let P be a point interior to square $ABCD$ so that $PA = a$, $PB = b$, $PC = c$, and $c^2 = a^2 + 2b^2$. Given only the lengths a, b, and c, and using only a compass and a straight-edge, construct a square congruent to square $ABCD$.