USAMTS Problems 2002

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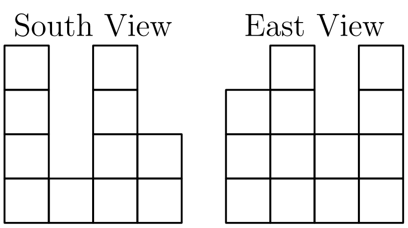
– Round 1

1 Some unit cubes are stacked atop a flat 4 by 4 square. The figures show views of the stacks from two different sides. Find the maximum and minimum number of cubes that could be in the stacks. Also give top views of a maximum arrangement and a minimum arrangement with each stack marked with its height.

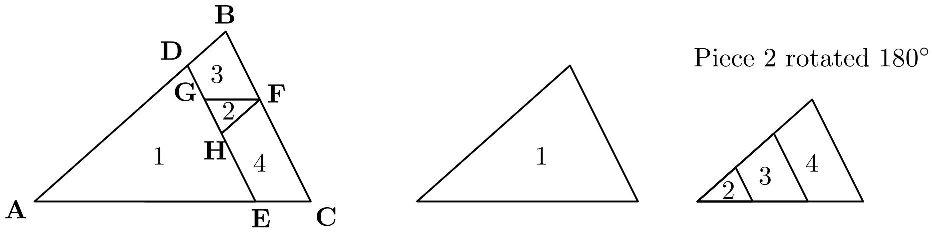


2 Find four distinct positive integers, a, b, c, and d, such that each of the four sums $a+b+c$, $a+b+d$,$a+c+d$, and $b+c+d$ is the square of an integer. Show that infinitely many quadruples (a,b,c,d) with this property can be created.

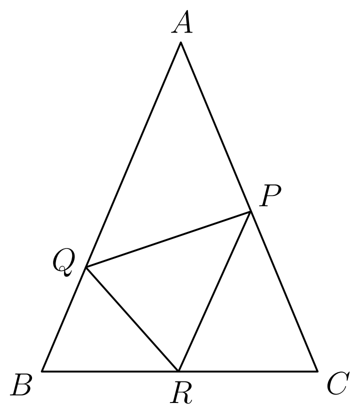
3 For a set of points in a plane, we construct the perpendicular bisectors of the line segments connecting every pair of those points and we count the number of points in which these perpendicular bisectors intersect each other. If we start with twelve points, the maximum possible number of intersection points is 1705. What is the maximum possible number of intersection points if we start with thirteen points?

4 A transposition of a vector is created by switching exactly two entries of the vector. For example, $(1,5,3,4,2,6,7)$ is a transposition of $(1,2,3,4,5,6,7)$. Find the vector X if $S=(0,0,1,1,0,1,1)$, $T=(0,0,1,1,1,1,0)$, $U=(1,0,1,0,1,1,0)$, and $V=(1,1,0,1,0,1,0)$ are all transpositions of X. Describe your method for finding X.

5 As illustrated below, we can dissect every triangle ABC into four pieces so that piece 1 is a triangle similar to the original triangle, while the other three pieces can be assembled into a triangle also similar to the original triangle. Determine the ratios of the sizes of the three triangles and verify that the construction works.



–	Round 2
1	Each member of the sequence $112002, 11210, 1121, 117, 46, 34, \ldots$ is obtained by adding five times the rightmost digit to the number formed by omitting that digit. Determine the billionth (10^9th) member of this sequence.
2	The integer 72 is the first of three consecutive integers 72, 73, and 74, that can each be expressed as the sum of the squares of two positive integers. The integers 72, 288, and 800 are the first three members of an infinite increasing sequence of integers with the above property. Find a function that generates the sequence and give the next three members.
3	An integer lattice point in the Cartesian plane is a point (x, y) where x and y are both integers. Suppose nine integer lattice points are chosen such that no three of them lie on the same line. Out of all 36 possible line segments between pairs of those nine points, some line segments may contain integer lattice points besides the original nine points. What is the minimum number of line segments that must contain an integer lattice point besides the original nine points? Prove your answer.
4	Let $f(n)$ be the number of ones that occur in the decimal representations of all the numbers from 1 to n. For example, this gives $f(8) = 1$, $f(9) = 1$, $f(10) = 2$, $f(11) = 4$, and $f(12) = 5$. Determine the value of $f(10^{100})$.
5	For an isosceles triangle ABC where $AB = AC$, it is possible to construct, using only compass and straightedge, an isosceles triangle PQR where $PQ = PR$ such that triangle PQR is similar to triangle ABC, point P is in the interior of line segment AC, point Q is in the interior of line segment AB, and point R is in the interior of line segment BC. Describe one method of performing such a construction. Your method should work on every isosceles triangle ABC, except that you may choose an upper limit or lower limit on the size of angle BAC.



– Round 3

1 The integer n, between 10000 and 99999, is $abcde$ when written in decimal notation. The digit a is the remainder when n is divided by 2, the digit b is the remainder when n is divided by 3, the digit c is the remainder when n is divided by 4, the digit d is the remainder when n is divied by 5, and the digit e is the reminader when n is divided by 6. Find n.

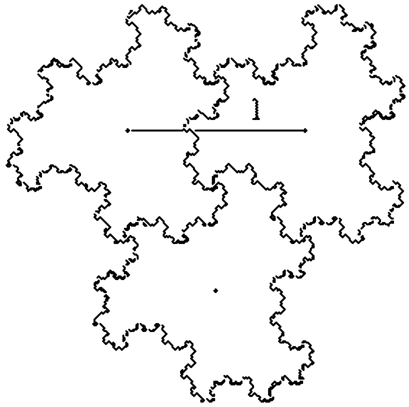
2 Given positive integers p, u, and v such that $u^2+2v^2=p$, determine, in terms of u and v, integers m and n such that $3m^2-2mn+3n^2=24p$. (It is known that if p is any prime number congruent to 1 or 3 modulo 8, then we can find integers u and v such that $u^2+2v^2=p$)

3 Determine, with proof, the rational number $\frac{m}{n}$ that equals

$$\frac{1}{1\sqrt{2}+2\sqrt{1}}+\frac{1}{2\sqrt{3}+3\sqrt{2}}+\frac{1}{3\sqrt{4}+4\sqrt{3}}+\dots+\frac{1}{4012008\sqrt{4012009}+4012009\sqrt{4012008}}$$

4 The vertices of a cube have coordinates $(0,0,0)$, $(0,0,4)$, $(0,4,0)$, $(0,4,4)$, $(4,0,0)$,$(4,0,4)$, $(4,4,0)$, and $(4,4,4)$. A plane cuts the edges of this cube at the points $(0,2,0)$, $(1,0,0)$, $(1,4,4)$, and two other points. Find the coordinates of the other two points.

5 A fudgeflake is a planar fractal figure with a 120° rotational symmetry such that three identical fudgeflakes in the same orientation fit together without gaps to form a larger fudgeflake with its orientation 30° clockwise of the smaller fudgeflakes' orientation, as shown below. If the distance between the centers of the original three fudgeflakes is 1, what is the area of one of those three fudgeflakes? Justify your answer.



– Round 4

1 The sequence of letters **TAGC** is written in succession 55 times on a strip, as shown below. The strip is to be cut into segments between letters, leaving strings of letters on each segment, which we call words. For example, a cut after the first G, after the second T, and after the second C would yield the words **TAG**, **CT** and **AGC**. At most how many distinct words could be found if the entire strip were cut? Justify your answer.

TAGCTAGCTAG . . . **CTAGC**

2 We define the number s as

$$s = \sum_{i=1}^{\infty} \frac{1}{10^i - 1} = \frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \ldots = 0.12232424...$$

We can determine the nth digit right of the decimal point of s without summing the entire infinite series because after summing the first n terms of the series, the rest of the series sums to less than $\frac{2}{10^{n+1}}$. Determine the smallest prime number p for which the pth digit right of the decimal point of s is greater than 2. Justify your answer.

3 Find the real-numbered solution to the equation below and demonstrate that it is unique.

$$\frac{36}{\sqrt{x}} + \frac{9}{\sqrt{y}} = 42 - 9\sqrt{x} - \sqrt{y}$$

4 Two overlapping triangles could divide a plane into up to eight regions, and three overlapping triangles could divide the plane into up to twenty regions. Find, with proof, the maximum number of regions into which six overlapping triangles could divide the plane. Describe or draw an arrangement of six triangles that divides the plane into that many regions.

5 Prove that if the cross-section of a cube cut by a plane is a pentagon, as shown in the figure below, then there are two adjacent sides of the pentagon such that the sum of the lengths of those two sides is greater than the sum of the lengths of the other three sides. (For ease of grading, please use the names of the points from the figure below in your solution.)