Belarusian National Olympiad 2007

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–	Day 1
1	Find all polynomials with degree $\leq n$ and nonnegative coefficients, such that $P(x)P(\frac{1}{x}) \leq P(1)^2$ for every positive x
2	Circles S_1 and S_2 with centers O_1 and O_2, respectively, pass through the centers of each other. Let A be one of their intersection points. Two points M_1 and M_2 begin to move simultaneously starting from A. Point M_1 moves along S_1 and point M_2 moves along S_2. Both points move in clockwise direction and have the same linear velocity v. (a) Prove that all triangles AM_1M_2 are equilateral. (b) Determine the trajectory of the movement of the center of the triangle AM_1M_2 and find its linear velocity.
3	Given a $2n \times 2m$ table $(m, n \in \mathbb{N})$ with one of two signs + or - in each of its cells. A union of all the cells of some row and some column is called a cross. The cell on the intersectin of this row and this column is called the center of the cross. The following procedure we call a transformation of the table: we mark all cells which contain and then, in turn, we replace the signs in all cells of the crosses which centers are marked by the opposite signs. (It is easy to see that the order of the choice of the crosses doesnt matter.) We call a table attainable if it can be obtained from some table applying such transformations one time. Find the number of all attainable tables.
4	Each point of a circle is painted in one of the N colors ($N \geq 2$). Prove that there exists an inscribed trapezoid such that all its vertices are painted the same color.
–	Day 2
5	Let O be the intersection point of the diagonals of the convex quadrilateral $ABCD$, $AO = CO$. Points P and Q are marked on the segments AO and CO, respectively, such that $PO = OQ$. Let N and K be the intersection points of the sides AB, CD, and the lines DP and BQ respectively. Prove that the points N, O, and K are colinear.
6	Let a be the sum and b the product of the real roots of the equation $x^4 - x^3 - 1 = 0$ Prove that $b < -\frac{11}{10}$ and $a > \frac{6}{11}$.

7 Find solution in positive integers :

$$n^5 + n^4 = 7^m - 1$$

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8 Let (m, n) be a pair of positive integers.
(a) Prove that the set of all positive integers can be partitioned into four pairwise disjoint nonempty subsets such that none of them has two numbers with absolute value of their difference equal to either m, n, or $m + n$.
(b) Find all pairs (m, n) such that the set of all positive integers can not be partitioned into three pairwise disjoint nonempty subsets satisfying the above condition.
