USAMTS Problems 2004

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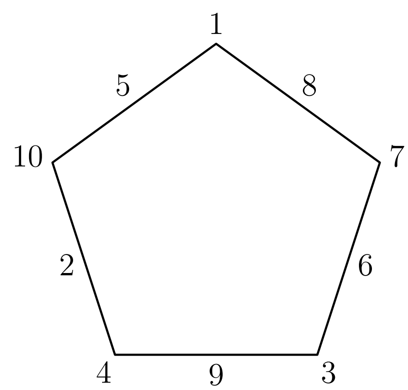
– Round 1

1 The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum 16. Find, with proof, the smallest possible value for a sum and give an example of an arrangement with that sum.



2 For the equation

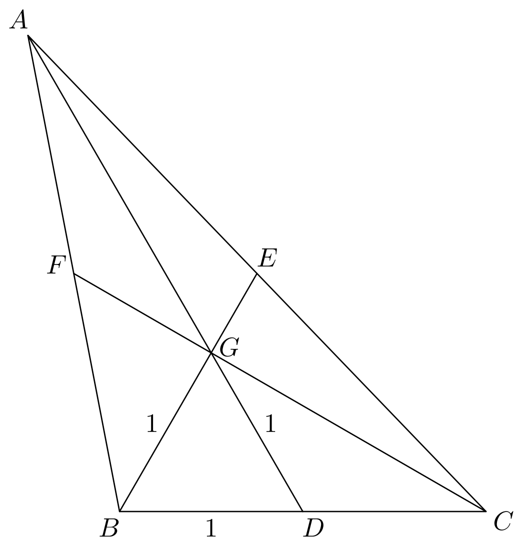
$$(3x^2 + y^2 - 4y - 17)^3 - (2x^2 + 2y^2 - 4y - 6)^3 = (x^2 - y^2 - 11)^3,$$

determine its solutions (x, y) where both x and y are integers. Prove that your answer lists all the integer solutions.

3 Given that $5r + 4s + 3t + 6u = 100$, where $r \geq s \geq t \geq u \geq 0.$ are real numbers, find, with proof, the maximum and minimum possible values of $r + s + t + u$.

4 The interior angles of a convex polygon form an arithmetic progression with a common difference of 4°. Determine the number of sides of the polygon if its largest interior angle is 172°.

5 Point G is where the medians of the triangle ABC intersect and point D is the midpoint of side BC. The triangle BDG is equilateral with side length 1. Determine the lengths, AB, BC, and CA, of the sides of triangle ABC.

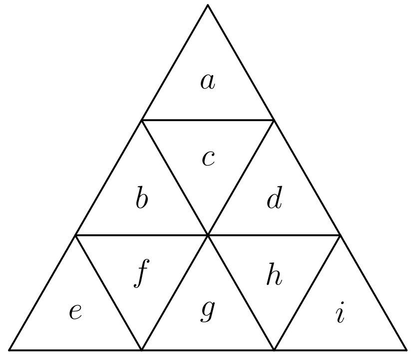


– Round 2

1 The numbers 1 through 9 can be arranged in the triangles labeled a through i illustrated below so that the numbers in each of the 2×2 triangles sum to the value n; that is

$$a+b+c+d=b+e+f+g=d+g+h+i=n.$$

For each possible sum n, show an arrangement, labeled with the sum as shown below. Prove that there are no possible arrangements for any other values of n.



2 Call a number $a-b\sqrt{2}$ with a and b both positive integers *tiny* if it is closer to zero than any number $c-d\sqrt{2}$ such that c and d are positive integers with $c<a$ and $d<b$. Three numbers which are tiny are $1-\sqrt{2}$, $3-2\sqrt{2}$, and $7-5\sqrt{2}$. Without using any calculator or computer,

prove whether or not each of the following is tiny:

$$(a)\ 58 - 41\sqrt{2}, \qquad (b)\ 99 - 70\sqrt{2}.$$

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3 A set is $reciprocally\ whole$ if its elements are distinct integers greater than 1 and the sum of the reciprocals of all these elements is exactly 1. Find a set S, as small as possible, that contains two reciprocally whole subsets, I and J, which are distinct, but not necessarily disjoint (meaning they may share elements, but they may not be the same subset). Prove that no set with fewer elements than S can contain two reciprocally whole subsets.

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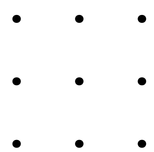
4 How many quadrilaterals in the plane have four of the nine points $(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0$ as vertices? Do count both concave and convex quadrilaterals, but do not count figures where two sides cross each other or where a vertex angle is $180°$. Rigorously verify that no quadrilateral was skipped or counted more than once.



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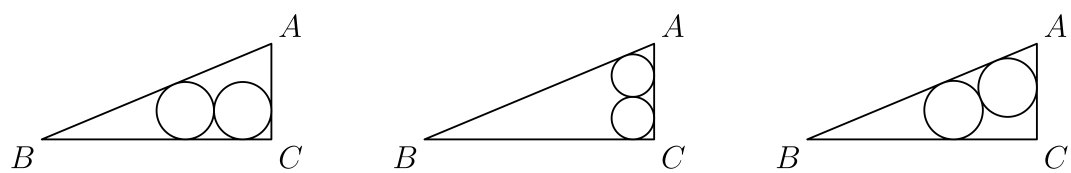
5 Two circles of equal radius can tightly fit inside right triangle ABC, which has $AB = 13$, $BC = 12$, and $CA = 5$, in the three positions illustrated below. Determine the radii of the circles in each case.



Case (i) Case (ii) Case (iii)

– Round 3

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1 Given two integers x and y, let $(x\|y)$ denote the *concatenation* of x by y, which is obtained by appending the digits of y onto the end of x. For example, if $x = 218$ and $y = 392$, then $(x\|y) = 218392$.
(a) Find 3-digit integers x and y such that $6(x\|y) = (y\|x)$.
(b) Find 9-digit integers x and y such that $6(x\|y) = (y\|x)$.

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2 Find three isosceles triangles, no two of which are congruent, with integer sides, such that each triangle's area is numerically equal to 6 times its perimeter.

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3 Define the recursive sequence $1, 4, 13, \ldots$ by $s_1 = 1$ and $s_{n+1} = 3s_n + 1$ for all positive integers n. The element $s_{18} = 193710244$ ends in two identical digits. Prove that all the elements in the sequence that end in two or more identical digits come in groups of three consecutive elements that have the same number of identical digits at the end.

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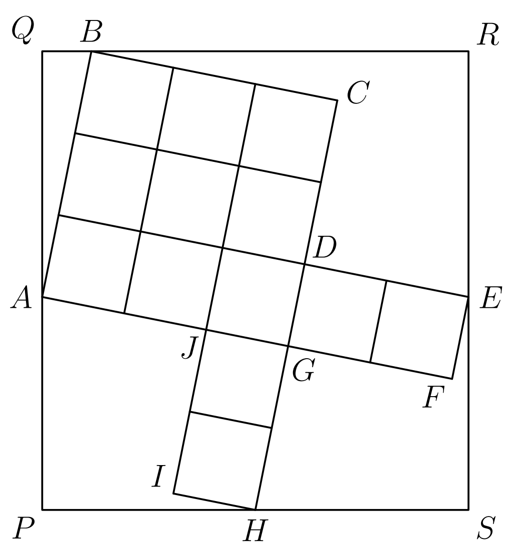
4 Region $ABCDEFGHIJ$ consists of 13 equal squares and is inscribed in rectangle $PQRS$ with A on $\overline{PQ}$, B on $\overline{QR}$, E on $\overline{RS}$, and H on $\overline{SP}$, as shown in the figure on the right. Given that $PQ = 28$ and $QR = 26$, determine, with proof, the area of region $ABCDEFGHIJ$.



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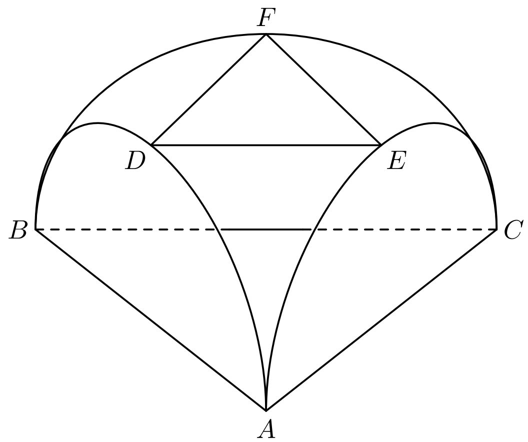
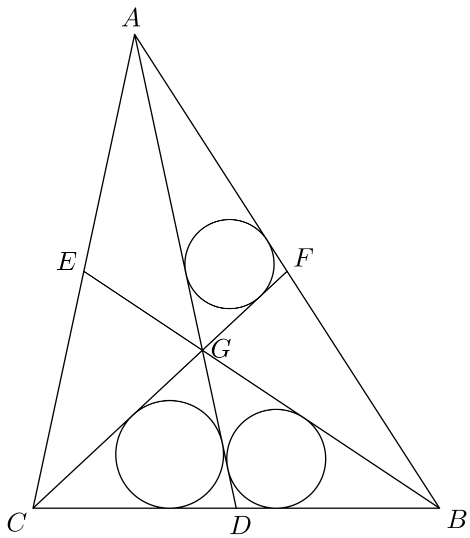
5 Consider an isosceles triangle ABC with side lengths $AB = AC = 10\sqrt{2}$ and $BC = 10\sqrt{3}$. Construct semicircles P, Q, and R with diameters AB, AC, BC respectively, such that the plane of each semicircle is perpendicular to the plane of ABC, and all semicircles are on the same side of plane ABC as shown. There exists a plane above triangle ABC that is tangent to all three semicircles P, Q, R at the points D, E, and F respectively, as shown in the diagram. Calculate, with proof, the area of triangle DEF.



–	Round 4
1	Determine with proof the number of positive integers n such that a convex regular polygon with n sides has interior angles whose measures, in degrees, are integers.
2	Find positive integers a, b, and c such that $$\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}}.$$ Prove that your solution is correct. (Warning: numerical approximations of the values do not constitute a proof.)
3	Find, with proof, a polynomial $f(x,y,z)$ in three variables, with integer coefficients, such that for all a,b,c the sign of $f(a,b,c)$ (that is, positive, negative, or zero) is the same as the sign of $a+b\sqrt[3]{2}+c\sqrt[3]{4}$.
4	Find, with proof, all integers n such that there is a solution in nonnegative real numbers (x,y,z) to the system of equations $$2x^2+3y^2+6z^2=n \text{ and } 3x+4y+5z=23.$$
5	Medians AD, BE, and CF of triangle ABC meet at G as shown. Six small triangles, each with vertex at G, are formed. We draw the circles inscribed in triangles AFG, BDG, and CDG as shown. Prove that if these three circles are all congruent, then ABC is equilateral.


A
E
F
G
C
D
B
