USAMTS Problems 2005

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– Round 1

1 An increasing arithmetic sequence with infinitely many terms is determined as follows. A single die is thrown and the number that appears is taken as the first term. The die is thrown again and the second number that appears is taken as the common difference between each pair of consecutive terms. Determine with proof how many of the 36 possible sequences formed in this way contain at least one perfect square.

2 George has six ropes. He chooses two of the twelve loose ends at random (possibly from the same rope), and ties them together, leaving ten loose ends. He again chooses two loose ends at random and joins them, and so on, until there are no loose ends. Find, with proof, the expected value of the number of loops George ends up with.

3 Let r be a nonzero real number. The values of z which satisfy the equation

$$r^4z^4 + (10r^6 - 2r^2)z^2 - 16r^5z + (9r^8 + 10r^4 + 1) = 0$$

are plotted on the complex plane (i.e. using the real part of each root as the x-coordinate and the imaginary part as the y-coordinate). Show that the area of the convex quadrilateral with these points as vertices is independent of r, and find this area.

4 Homer gives mathematicians Patty and Selma each a different integer, not known to the other or to you. Homer tells them, within each others hearing, that the number given to Patty is the product ab of the positive integers a and b, and that the number given to Selma is the sum $a+b$ of the same numbers a and b, where $b > a > 1$. He doesnt, however, tell Patty or Selma the numbers a and b. The following (honest) conversation then takes place:
Patty: I cant tell what numbers a and b are.
Selma: I knew before that you couldnt tell.
Patty: In that case, I now know what a and b are.
Selma: Now I also know what a and b are.
Supposing that Homer tells you (but neither Patty nor Selma) that neither a nor b is greater than 20, find a and b, and prove your answer can result in the conversation above.

5 Given triangle ABC, let M be the midpoint of side AB and N be the midpoint of side AC. A circle is inscribed inside quadrilateral $NMBC$, tangent to all four sides, and that circle touches MN at point X. The circle inscribed in triangle AMN touches MN at point Y, with Y between X and N. If $XY = 1$ and $BC = 12$, find, with proof, the lengths of the sides AB and AC.

– Round 2

1 Below is a 4×4 grid. We wish to fill in the grid such that each row, each column, and each 2×2 square outlined by the double lines contains the digits 1 through 4. The first row has already been filled in. Find, with proof, the number of ways we can complete the rest of the grid.

1	2	3	4

2 Write the number

$$\frac{1}{\sqrt{2} - \sqrt[3]{2}}$$

as the sum of terms of the form 2^q, where q is rational. (For example, $2^1 + 2^{-1/3} + 2^{8/5}$ is a sum of this form.) Prove that your sum equals $1/(\sqrt{2} - \sqrt[3]{2})$.

3 An equilateral triangle is tiled with n^2 smaller congruent equilateral triangles such that there are n smaller triangles along each of the sides of the original triangle. For each of the small equilateral triangles, we randomly choose a vertex V of the triangle and draw an arc with that vertex as center connecting the midpoints of the two sides of the small triangle with V as an endpoint. Find, with proof, the expected value of the number of full circles formed, in terms of n.

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4 A teacher plays the game Duck-Goose-Goose with his class. The game is played as follows: All the students stand in a circle and the teacher walks around the circle. As he passes each student, he taps the student on the head and declares her a duck or a goose. Any student named a goose leaves the circle immediately. Starting with the first student, the teacher tags students in the pattern: duck, goose, goose, duck, goose, goose, etc., and continues around the circle (re-tagging some former ducks as geese) until only one student remains. This remaining student is the winner.

For instance, if there are 8 students, the game proceeds as follows: student 1 (duck), student 2 (goose), student 3 (goose), student 4 (duck), student 5 (goose), student 6 (goose), student 7 (duck), student 8 (goose), student 1 (goose), student 4 (duck), student 7 (goose) and student 4 is the winner. Find, with proof, all values of n with $n > 2$ such that if the circle starts with n students, then the nth student is the winner.

5 Given acute triangle $\triangle ABC$ in plane P, a point Q in space is defined such that $\angle AQB = \angle BQC = \angle CQA = 90^\circ$. Point X is the point in plane P such that QX is perpendicular to plane P. Given $\angle ABC = 40^\circ$ and $\angle ACB = 75^\circ$, find $\angle AXC$.

– Round 3

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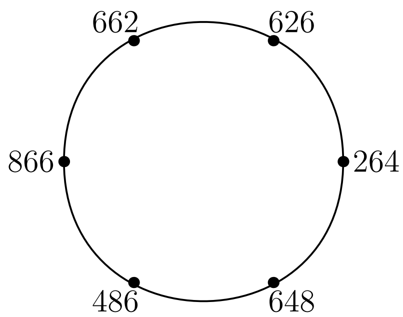
1 For a given positive integer n, we wish to construct a circle of six numbers as shown below so that the circle has the following properties:

(a) The six numbers are different three-digit numbers, none of whose digits is a 0.
(b) Going around the circle clockwise, the first two digits of each number are the last two digits, in the same order, of the previous number.
(c) All six numbers are divisible by n.

The example above shows a successful circle for $n = 2$. For each of $n = 3, 4, 5, 6, 7, 8, 9$, either construct a circle that satisfies these properties, or prove that it is impossible to do so.



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2 Anna writes a sequence of integers starting with the number 12. Each subsequent integer she writes is chosen randomly with equal chance from among the positive divisors of the previous integer (including the possibility of the integer itself). She keeps writing integers until she writes the integer 1 for the first time, and then she stops. One such sequence is

$$12, 6, 6, 3, 3, 3, 1.$$

What is the expected value of the number of terms in Annas sequence?

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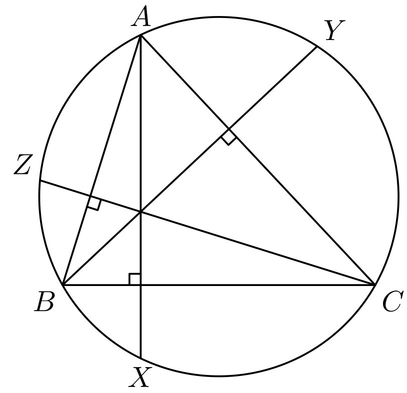
3 Points A, B, and C are on a circle such that $\triangle ABC$ is an acute triangle. X, Y, and Z are on the circle such that AX is perpendicular to BC at D, BY is perpendicular to AC at E, and CZ is perpendicular to AB at F. Find the value of

$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF},$$

and prove that this value is the same for all possible A, B, C on the circle such that $\triangle ABC$ is acute.



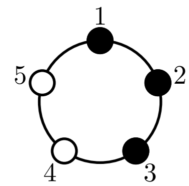
4 Find, with proof, all triples of real numbers (a, b, c) such that all four roots of the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + b$ are positive integers. (The four roots need not be distinct.)

5 Lisa and Bart are playing a game. A round table has n lights evenly spaced around its circumference. Some of the lights are on and some of them off; the initial configuration is random. Lisa wins if she can get all of the lights turned on; Bart wins if he can prevent this from happening.

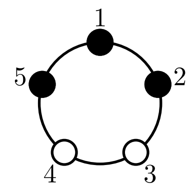
On each turn, Lisa chooses the positions at which to flip the lights, but before the lights are flipped, Bart, knowing Lisas choices, can rotate the table to any position that he chooses (or he can leave the table as is). Then the lights in the positions that Lisa chose are flipped: those that are off are turned on and those that are on are turned off.

Here is an example turn for $n = 5$ (a white circle indicates a light that is on, and a black circle indicates a light that is off):

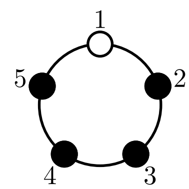
Initial Position.



Lisa says “1,3,4.”
Bart rotates the table one position counterclockwise.



Lights in positions 1,3,4 are flipped.



Lisa can take as many turns as she needs to win, or she can give up if it becomes clear to her that Bart can prevent her from winning.

(a) Show that if $n = 7$ and initially at least one light is on and at least one light is off, then Bart can always prevent Lisa from winning.

(b) Show that if $n = 8$, then Lisa can always win in at most 8 turns.

– Round 4

1 $\overline{AB}$ is a diameter of circle C_1. Point P is on C_1 such that $AP > BP$. Circle C_2 is centered at P with radius PB. The extension of $\overline{AP}$ past P meets C_2 at Q. Circle C_3 is centered at A and is externally tangent to C_2. Circle C_4 passes through A, Q, and R. Find, with proof, the ratio between the area of C_4 and the area of C_1, and show that this ratio is the same for all points P on C_1 such that $AP > BP$.

2 *Centered hexagonal numbers* are the numbers of dots used to create hexagonal arrays of dots. The first four centered hexagonal numbers are 1, 7, 19, and 37 as shown below:

1 7 19 37

Centered Hexagonal Numbers

Consider an arithmetic sequence 1, a, b and a geometric sequence 1,c,d, where a,b,c, and d are all positive integers and $a+b=c+d$. Prove that each centered hexagonal number is a possible value of a, and prove that each possible value of a is a centered hexagonal number.

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3 We play a game. The pot starts at $\$0$. On every turn, you flip a fair coin. If you flip heads, I add $\$100$ to the pot. If you flip tails, I take all of the money out of the pot, and you are assessed a ”strike”. You can stop the game before any flip and collect the contents of the pot, but if you get 3 strikes, the game is over and you win nothing. Find, with proof, the expected value of your winnings if you follow an optimal strategy.

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4 Find, with proof, all irrational numbers x such that both x^3-6x and x^4-8x^2 are rational.

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5 Sphere S is inscribed in cone C. The height of C equals its radius, and both equal $12+12\sqrt{2}$. Let the vertex of the cone be A and the center of the sphere be B. Plane P is tangent to S and intersects $\overline{AB}$. X is the point on the intersection of P and C closest to A. Given that $AX=6$, find the area of the region of P enclosed by the intersection of C and P.
