Spain Mathematical Olympiad 2007

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– Session 1

Problem 1 Let a_0, a_1, a_2, a_3, a_4 be five positive numbers in the arithmetic progression with a difference d. Prove that $a_2^3 \leq \frac{1}{10}(a_0^3 + 4a_1^3 + 4a_3^3 + a_4^3)$.

Problem 2 Determine all the possible non-negative integer values that are able to satisfy the expression: $\frac{(m^2+mn+n^2)}{(mn-1)}$
if m and n are non-negative integers such that $mn \neq 1$.

Problem 3 O is the circumcenter of triangle ABC. The bisector from A intersects the opposite side in point P. Prove that the following is satisfied:

$$AP^2 + OA^2 - OP^2 = bc.$$

– Session 2

Problem 4 What are the positive integer numbers that we are able to obtain in 2007 distinct ways, when the sum is at least out of two positive consecutive integers? What is the smallest of all of them?
Example: the number 9 is written in exactly two such distinct ways: $9 = 4 + 5$ $9 = 2 + 3 + 4$.

Problem 5 Let $a \neq 1$ and be a real positive number and n be an integer greater than 1. Demonstrate that $n^2 < \frac{(a^n+a^{-n}-2)}{(a+a^{-1}-2)}$.

Problem 6 Given a halfcircle of diameter $AB = 2R$, consider a chord CD of length c. Let E be the intersection of AC with BD and F the inersection of AD with BC.
Prove that the segment EF has a constant length and direction when varying the chord CD about the halfcircle.