USAMTS Problems 2013

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by El_Ectric

–	Round 1
–	October 15th

1 Alex is trying to open a lock whose code is a sequence that is three letters long, with each of the letters being one of A, B or C, possibly repeated. The lock has three buttons, labeled A, B and C. When the most recent 3 button-presses form the code, the lock opens. What is the minimum number of total button presses Alex needs to guarantee opening the lock?

2 In the 5×6 grid shown, fill in all of the grid cells with the digits 0–9 so that the following conditions are satisfied:
- Each digit gets used exactly 3 times.
- No digit is greater than the digit directly above it.
- In any four cells that form a 2×2 subgrid, the sum of the four digits must be a multiple of 3.
You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that works. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

				7	
	8				6
		2	4		
5				1	
	3				

3 An infinite sequence of positive real numbers $a_1, a_2, a_3, \dots$ is called *territorial* if for all positive integers i, j with $i < j$, we have $|a_i - a_j| \geq \frac{1}{j}$. Can we find a territorial sequence $a_1, a_2, a_3, \dots$ for which there exists a real number c with $a_i < c$ for all i?

4 Bunbury the bunny is hopping on the positive integers. First, he is told a positive integer n. Then Bunbury chooses positive integers a, d and hops on all of the spaces $a, a+d, a+2d, \dots, a+2013d$. However, Bunbury must make these choices so that the number of every space that he hops on is less than n and relatively prime to n.

A positive integer n is called *bunny-unfriendly* if, when given that n, Bunbury is unable to find positive integers a, d that allow him to perform the hops he wants. Find the maximum bunny-unfriendly integer, or prove that no such maximum exists.

5 Niki and Kyle play a triangle game. Niki first draws $\triangle ABC$ with area 1, and Kyle picks a point X inside $\triangle ABC$. Niki then draws segments $\overline{DG}$, $\overline{EH}$, and $\overline{FI}$, all through X, such that D and E are on $\overline{BC}$, F and G are on $\overline{AC}$, and H and I are on $\overline{AB}$. The ten points must all be distinct. Finally, let S be the sum of the areas of triangles DEX, FGX, and HIX. Kyle earns S points, and Niki earns $1-S$ points. If both players play optimally to maximize the amount of points they get, who will win and by how much?

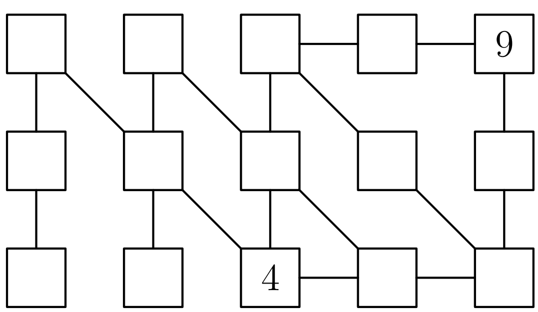
– Round 2

– November 18th

1 In the 3×5 grid shown, fill in each empty box with a two-digit positive integer such that:
-no number appears in more than one box, and

- for each of the 9 lines in the grid consisting of three boxes connected by line segments, the box in the middle of the line contains the least common multiple of the numbers in the two boxes on the line.

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



2 Let $ABCD$ be a quadrilateral with $\overline{AB}\parallel\overline{CD}$, $AB=16$, $CD=12$, and $BC<AD$. A circle with diameter 12 is inside of $ABCD$ and tangent to all four sides. Find BC.

3 For each positive integer $n\ge 2$, find a polynomial $P_n(x)$ with rational coefficients such that $P_n(\sqrt[n]{2})=\dfrac{1}{1+\sqrt[n]{2}}$. (Note that $\sqrt[n]{2}$ denotes the positive n^{th} root of 2.)

4 An infinite sequence of real numbers $a_1,a_2,a_3,\dots$ is called *spooky* if $a_1=1$ and for all integers

$n > 1$,

$$\begin{array}{ccccccccccccc} na_1 & + & (n-1)a_2 & + & (n-2)a_3 & + & \dots & + & 2a_{n-1} & + & a_n & < & 0, \\ n^2a_1 & + & (n-1)^2a_2 & + & (n-2)^2a_3 & + & \dots & + & 2^2a_{n-1} & + & a_n & > & 0. \end{array}$$

Given any spooky sequence $a_1, a_2, a_3, \dots$, prove that

$$2013^3a_1 + 2012^3a_2 + 2011^3a_3 + \cdots + 2^3a_{2012} + a_{2013} < 12345.$$

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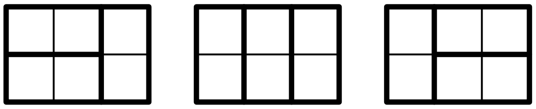
5 Let S be a planar region. A *domino-tiling* of S is a partition of S into 1×2 rectangles. (For example, a 2×3 rectangle has exactly 3 domino-tilings, as shown below.)



The rectangles in the partition of S are called *dominoes*.
(a) For any given positive integer n, find a region S_n with area at most $2n$ that has exactly n domino-tilings.

(b) Find a region T with area less than 50000 that has exactly 100002013 domino-tilings.

– Round 3

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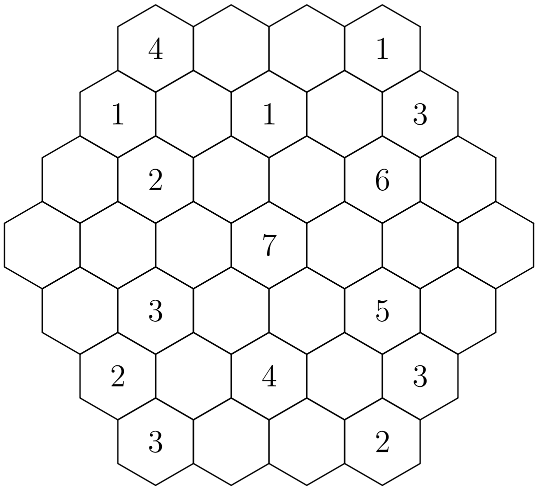
1 In the grid shown, fill in each empty space with a number, such that after the grid is completely filled in, the number in each space is equal to the smallest positive integer that does not appear in any of the touching spaces. (A pair of spaces is considered to touch if they both share a vertex.)

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



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2 Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers such that $a_k a_{k+2} = a_{k+1} + 1$ for all positive integers k. If a_1 and a_2 are positive integers, find the maximum possible value of a_{2014}.

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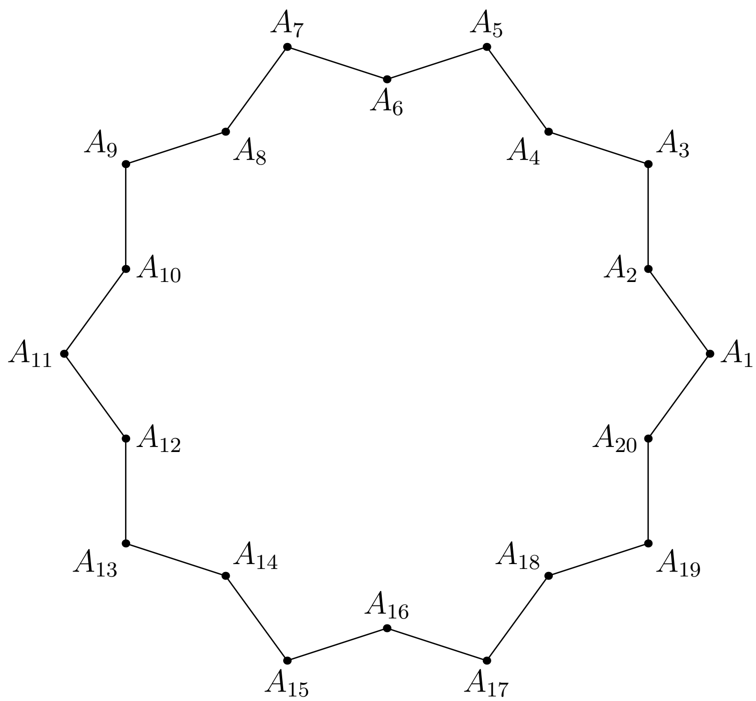
3 Let $A_1A_2A_3 \ldots A_{20}$ be a 20-sided polygon P in the plane, where all of the side lengths of P are equal, the interior angle at A_i measures 108 degrees for all odd i, and the interior angle A_i measures 216 degrees for all even i. Prove that the lines A_2A_8, A_4A_{10}, A_5A_{13}, A_6A_{16}, and A_7A_{19} all intersect at the same point.



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4 An infinite sequence $(a_0, a_1, a_2, \dots)$ of positive integers is called a *ribbon* if the sum of any eight consecutive terms is at most 16; that is, for all $i \geq 0$,

$$a_i + a_{i+1} + \cdots + a_{i+7} \leq 16.$$

A positive integer m is called a *cut size* if every ribbon contains a set of consecutive elements that sum to m; that is, given any ribbon $(a_0, a_1, a_2, \dots)$, there exist nonnegative integers $k \leq l$ such that

$$a_k + a_{k+1} + \cdots + a_l = m.$$

Find, with proof, all cut sizes, or prove that none exist.

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5 For any positive integer $b \geq 2$, we write the base-b numbers as follows:

$$(d_k d_{k-1} \dots d_0)_b = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_1 b^1 + d_0 b^0,$$

where each digit d_i is a member of the set $S = \{0, 1, 2, \dots, b-1\}$ and either $d_k \neq 0$ or $k = 0$. There is a unique way to write any nonnegative integer in the above form. If we select the digits from a di fferent set S instead, we may obtain new representations of all positive integers or, in some cases, all integers. For example, if $b = 3$ and the digits are selected from $S = \{-1, 0, 1\}$, we obtain a way to uniquely represent all integers, known as a *balanced ternary* representation. As further examples, the balanced ternary representation of numbers 5, -3, and 25 are:

$$5 = (1\ {-1}\ {-1})_3, \qquad -3 = ({-1}\ 0)_3, \qquad 25 = (1\ 0\ {-1}\ 1)_3.$$

However, not all digit sets can represent all integers. If $b = 3$ and $S = \{-2, 0, 2\}$, then no odd number can be represented. Also, if $b = 3$ and $S = \{0, 1, 2\}$ as in the usual base-3 representation, then no negative number can be represented.

Given a set S of four integers, one of which is 0, call S a ***4-basis*** if every integer n has at least one representation in the form

$$n = (d_k d_{k-1} \dots d_0)_4 = d_k 4^k + d_{k-1} 4^{k-1} + \cdots + d_1 4^1 + d_0 4^0,$$

where $d_k, d_{k-1}, \dots, d_0$ are all elements of S and either $d_k \neq 0$ or $k = 0$.

-Show that there are infinitely many integers a such that $\{-1, 0, 1, 4a + 2\}$ is not a 4-basis.
-Show that there are infinitely many integers a such that $\{-1, 0, 1, 4a + 2\}$ is a 4-basis.