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Day 1

1 Let $ABCD$ be a tetrahedron with $\angle BAD = 60^{\circ}$, $\angle BAC = 40^{\circ}$, $\angle ABD = 80^{\circ}$, $\angle ABC = 70^{\circ}$. Prove that the lines AB and CD are perpendicular.

2 Let $s \ge 3$ be a given integer. A sequence K_n of circles and a sequence W_n of convex s-gons satisfy:

$$K_n \supset W_n \supset K_{n+1}$$

for all $n = 1, 2, ...$
Prove that the sequence of the radii of the circles K_n converges to zero.

3 Consider the set $A = \{0, 1, 2, ..., 2^{2n} - 1\}$. The function $f : A \to A$ is given by: $f(x_0 + 2x_1 + 2^2x_2 + ... + 2^{2n-1}x_{2n-1}) =(1 - x_0) + 2x_1 + 2^2(1 - x_2) + 2^3x_3 + ... + 2^{2n-1}x_{2n-1}$
for every $0 - 1$ sequence $(x_0, x_1, ..., x_{2n-1})$. Show that if $a_1, a_2, ..., a_9$ are consecutive terms of an arithmetic progression, then the sequence $f(a_1), f(a_2), ..., f(a_9)$ is not increasing.

Day 2

1 A function $h : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies $h(ax) = bh(x)$ for all x, where a and b are given positive numbers and $0 \neq |a| \neq 1$. Suppose that $h'(0) \neq 0$ and the function h' is continuous at $x = 0$. Prove that $a = b$ and that there is a real number c such that $h(x) = cx$ for all x.

2 Show that for every convex polygon there is a circle passing through three consecutive vertices of the polygon and containing the entire polygon

3 Consider the polynomial $W(x) = (x - a)^kQ(x)$, where $a \neq 0$, Q is a nonzero polynomial, and k a natural number. Prove that W has at least $k + 1$ nonzero coefficients.