Finals 1988

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Day 1

1	The real numbers $x_1, x_2, ..., x_n$ belong to the interval $(0, 1)$ and satisfy $x_1 + x_2 + ... + x_n = m + r$, where m is an integer and $r \in [0, 1)$. Show that $x_1^2 + x_2^2 + ... + x_n^2 \leq m + r^2$.
2	For a permutation $P = (p_1, p_2, ..., p_n)$ of $(1, 2, ..., n)$ define $X(P)$ as the number of j such that $p_i < p_j$ for every $i < j$. What is the expected value of $X(P)$ if each permutation is equally likely?
3	W is a polygon which has a center of symmetry S such that if P belongs to W, then so does P', where S is the midpoint of PP'. Show that there is a parallelogram V containing W such that the midpoint of each side of V lies on the border of W.

Day 2

1	d is a positive integer and $f : [0, d] \to \mathbb{R}$ is a continuous function with $f(0) = f(d)$. Show that there exists $x \in [0, d - 1]$ such that $f(x) = f(x + 1)$.
2	The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = a_2 = a_3 = 1$, $a_{n+3} = a_{n+2}a_{n+1} + a_n$. Show that for any positive integer r we can find s such that a_s is a multiple of r.
3	Find the largest possible volume for a tetrahedron which lies inside a hemisphere of radius 1.