Finals 1989

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Day 1

1 An even number of politicians are sitting at a round table. After a break, they come back and sit down again in arbitrary places. Show that there must be two people with the same number of people sitting between them as before the break..

Additional problem:
Solve the problem when the number of people is in a form $6k+3$.

2 k_1, k_2, k_3 are three circles. k_2 and k_3 touch externally at P, k_3 and k_1 touch externally at Q, and k_1 and k_2 touch externally at R. The line PQ meets k_1 again at S, the line PR meets k_1 again at T. The line RS meets k_2 again at U, and the line QT meets k_3 again at V. Show that P, U, V are collinear.

3 The edges of a cube are labeled from 1 to 12. Show that there must exist at least eight triples (i, j, k) with $1 \le i < j < k \le 12$ so that the edges i, j, k are consecutive edges of a path. Also show that there exists labeling in which we cannot find nine such triples.

Day 2

1 n, k are positive integers. A_0 is the set $\{1, 2, ..., n\}$. A_i is a randomly chosen subset of A_{i-1} (with each subset having equal probability). Show that the expected number of elements of A_k is $\frac{n}{2^k}$

2 Three circles of radius a are drawn on the surface of a sphere of radius r. Each pair of circles touches externally and the three circles all lie in one hemisphere. Find the radius of a circle on the surface of the sphere which touches all three circles.

3 Show that for positive reals a, b, c, d we have

$$\left(\frac{ab+ac+ad+bc+bd+cd}{6}\right)^3 \ge \left(\frac{abc+abd+acd+bcd}{4}\right)^2$$