Finals 1990

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Day 1

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1 Find all functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ that satisfy

$$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$$

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2 Let $x_1, x_2, ..., x_n$ be positive numbers. Prove that

$$\sum_{i=1}^{n}\frac{x_i^2}{x_i^2+x_{i+1}x_{i+2}}\leq n-1$$

Where $x_{n+1}=x_1$ and $x_{n+2}=x_2$.

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3 In a tournament, every two of the n players played exactly one match with each other (no draws). Prove that it is possible either
(i) to partition the league in two groups A and B such that everybody in A defeated everybody in B; or
(ii) to arrange all the players in a chain $x_1, x_2, ..., x_n, x_1$ in such a way that each player defeated his successor.

Day 2

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1 A triangle whose all sides have length not smaller than 1 is inscribed in a square of side length 1. Prove that the center of the square lies inside the triangle or on its boundary.

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2 Suppose that (a_n) is a sequence of positive integers such that $\lim_{n\to\infty}\frac{n}{a_n}=0$
Prove that there exists k such that there are at least 1990 perfect squares between $a_1+a_2+...+a_k$ and $a_1+a_2+...+a_{k+1}$.

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3 Prove that for all integers $n>2$,

$$3|\sum_{i=0}^{[n/3]}(-1)^iC_n^{3i}$$
