Mediterranean Mathematics Olympiad 2017

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by socrates

Problem 1 Let ABC be an equilateral triangle, and let P be some point in its circumcircle. Determine all positive integers n, for which the value of the sum $S_n(P) = |PA|^n + |PB|^n + |PC|^n$ is independent of the choice of point P.

Problem 2 Determine the smallest integer n for which there exist integers $x_1, \ldots, x_n$ and positive integers $a_1, \ldots, a_n$ so that

$$\begin{aligned} x_1 + \cdots + x_n &= 0, \\ a_1x_1 + \cdots + a_nx_n &> 0, \text{ and} \\ a_1^2x_1 + \cdots + a_n^2x_n &< 0. \end{aligned}$$

Problem 3 A set S of integers is Balearic, if there are two (not necessarily distinct) elements $s, s' \in S$ whose sum $s + s'$ is a power of two; otherwise it is called a non-Balearic set. Find an integer n such that $\{1, 2, \ldots, n\}$ contains a 99-element non-Balearic set, whereas all the 100-element subsets are Balearic.

Problem 4 Let x, y, z and a, b, c be positive real numbers with $a + b + c = 1$. Prove that

$$(x^2 + y^2 + z^2)\left(\frac{a^3}{x^2 + 2y^2} + \frac{b^3}{y^2 + 2z^2} + \frac{c^3}{z^2 + 2x^2}\right) \geq \frac{1}{9}.$$
