Finals 1991

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Day 1

1 Prove or disprove that there exist two tetrahedra T_1 and T_2 such that:
(i) the volume of T_1 is greater than that of T_2;
(ii) the area of any face of T_1 does not exceed the area of any face of T_2.

2 Let X be the set of all lattice points in the plane (points (x,y) with $x,y\in\mathbb{Z}$). A path of length n is a chain $(P_0,P_1,...,P_n)$ of points in X such that $P_{i-1}P_i=1$ for $i=1,...,n$. Let $F(n)$ be the number of distinct paths beginning in $P_0=(0,0)$ and ending in any point P_n on line $y=0$. Prove that $F(n)=\binom{2n}{n}$

3 Define

$$N=\sum_{k=1}^{60} e_k k^{k^k}$$

where $e_k\in\{-1,1\}$ for each k. Prove that N cannot be the fifth power of an integer.

Day 2

1 On the Cartesian plane consider the set V of all vectors with integer coordinates. Determine all functions $f:V\to\mathbb{R}$ satisfying the conditions:
(i) $f(v)=1$ for each of the four vectors $v\in V$ of unit length.
(ii) $f(v+w)=f(v)+f(w)$ for every two perpendicular vectors $v,w\in V$
(Zero vector is considered to be perpendicular to every vector).

2 Two noncongruent circles k_1 and k_2 are exterior to each other. Their common tangents intersect the line through their centers at points A and B. Let P be any point of k_1. Prove that there is a diameter of k_2 with one endpoint on line PA and the other on PB.

3 If x,y,z are real numbers satisfying $x^2+y^2+z^2=2$, prove the inequality

$$x+y+z\le 2+xyz$$

When does equality occur?