Finals 1992

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Day 1

1	Segments AC and BD meet at P, and $\|PA\| = \|PD\|$, $\|PB\| = \|PC\|$. O is the circumcenter of the triangle PAB. Show that OP and CD are perpendicular.
2	Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the set of positive rationals, such that $f(x+1) = f(x) + 1$ and $f(x^3) = f(x)^3$ for all x.
3	Show that for real numbers $x_1, x_2, ..., x_n$ we have: $$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x_i x_j}{i+j} \geq 0$$ When do we have equality?

Day 2

1	The functions $f_0, f_1, f_2, ...$ are defined on the reals by $f_0(x) = 8$ for all x, $f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)}$. For all n solve the equation $f_n(x) = 2x$.
2	The base of a regular pyramid is a regular $2n$-gon $A_1A_2...A_{2n}$. A sphere passing through the top vertex S of the pyramid cuts the edge SA_i at B_i (for $i = 1, 2, ..., 2n$). Show that $\sum\limits_{i=1}^{n} SB_{2i-1} = \sum\limits_{i=1}^{n} SB_{2i}$.
3	Show that $(k^3)!$ is divisible by $(k!)^{k^2+k+1}$.