

Finals 1993

www.artofproblemsolving.com/community/c4687

by Megus, Severius, grobber, Rushil

Day 1

- 1 Find all rational solutions to:

$$t^2 - w^2 + z^2 = 2xy$$

$$t^2 - y^2 + w^2 = 2xz$$

$$t^2 - w^2 + x^2 = 2yz.$$

- 2 A circle center O is inscribed in the quadrilateral $ABCD$. AB is parallel to and longer than CD and has midpoint M . The line OM meets CD at F . CD touches the circle at E . Show that $DE = CF$ iff $AB = 2CD$.

- 3 Denote $g(k)$ as the greatest odd divisor of k . Put $f(k) = \frac{k}{2} + \frac{k}{g(k)}$ for k even, and $2^{(k+1)/2}$ for k odd. Define the sequence x_1, x_2, x_3, \dots by $x_1 = 1, x_{n+1} = f(x_n)$. Find n such that $x_n = 800$.

Day 2

- 1 Let be given a convex polyhedron whose all faces are triangular. The vertices of the polyhedron are colored using three colors. Prove that the number of faces with vertices in all the three colors is even.

- 2 Find all real-valued functions f on the reals such that $f(-x) = -f(x), f(x+1) = f(x) + 1$ for all x , and $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$ for $x \neq 0$.

- 3 Find out whether it is possible to determine the volume of a tetrahedron knowing the areas of its faces and its circumradius.