Finals 1994

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Day 1

1 Find all triples (x, y, z) of positive rationals such that $x+y+z$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ and xyz are all integers.

2 Let be given two parallel lines k and l, and a circle not intersecting k. Consider a variable point A on the line k. The two tangents from this point A to the circle intersect the line l at B and C. Let m be the line through the point A and the midpoint of the segment BC. Prove that all the lines m (as A varies) have a common point.

3 k is a fixed positive integer. Let a_n be the number of maps f from the subsets of $\{1, 2, ..., n\}$ to $\{1, 2, ..., k\}$ such that for all subsets A, B of $\{1, 2, ..., n\}$ we have $f(A \cap B) = \min(f(A), f(B))$. Find $\lim_{n\to\infty} \sqrt[n]{a_n}$.

Day 2

1 m, n are relatively prime. We have three jugs which contain m, n and $m+n$ liters. Initially the largest jug is full of water. Show that for any k in $\{1, 2, ..., m+n\}$ we can get exactly k liters into one of the jugs.

2 A parallelopiped has vertices $A_1, A_2, ..., A_8$ and center O. Show that:

$$4\sum_{i=1}^{8} OA_i^2 \le \left(\sum_{i=1}^{8} OA_i\right)^2$$

3 The distinct reals $x_1, x_2, ..., x_n$,($n > 3$) satisfy $\sum_{i=1}^n x_i = 0$, $\sum_{i=1}^n x_i^2 = 1$. Show that four of the numbers a, b, c, d must satisfy:

$$a+b+c+nabc \le \sum_{i=1}^{n} x_i^3 \le a+b+d+nabd$$

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