Finals 1996
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Day 1

1 Find all pairs (n, r) with n a positive integer and r a real such that $2x^2+2x+1$ divides $(x+1)^n-r$.

2 Let P be a point inside a triangle ABC such that $\angle PBC = \angle PCA < \angle PAB$. The line PB meets the circumcircle of triangle ABC at a point E (apart from B). The line CE meets the circumcircle of triangle APE at a point F (apart from E). Show that the ratio $\frac{|APEF|}{|ABP|}$ does not depend on the point P, where the notation $|P_1P_2...P_n|$ stands for the area of an arbitrary polygon $P_1P_2...P_n$.

3 a_i, x_i are positive reals such that $a_1 + a_2 + ... + a_n = x_1 + x_2 + ... + x_n = 1$. Show that

$$2\sum_{i<j} x_ix_j \leq \frac{n-2}{n-1} + \sum \frac{a_ix_i^2}{1-a_i}$$

When do we have equality?

Day 2

1 $ABCD$ is a tetrahedron with $\angle BAC = \angle ACD$ and $\angle ABD = \angle BDC$. Show that $AB = CD$.

2 Let $p(k)$ be the smallest prime not dividing k. Put $q(k) = 1$ if $p(k) = 2$, or the product of all primes $< p(k)$ if $p(k) > 2$. Define the sequence $x_0, x_1, x_2, ...$ by $x_0 = 1$, $x_{n+1} = \frac{x_np(x_n)}{q(x_n)}$. Find all n such that $x_n = 111111$

3 From the set of all permutations f of $\{1, 2, ..., n\}$ that satisfy the condition: $f(i) \geq i-1$ $i = 1, ..., n$
one is chosen uniformly at random. Let p_n be the probability that the chosen permutation f satisfies $f(i) \leq i+1$ $i = 1, ..., n$
Find all natural numbers n such that $p_n > \frac{1}{3}$.