Finals 1997

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Day 1

1 The positive integers $x_1, x_2, ..., x_7$ satisfy $x_6 = 144$, $x_{n+3} = x_{n+2}(x_{n+1} + x_n)$ for $n = 1, 2, 3, 4$. Find x_7.

2 Find all real solutions to:

$$\begin{aligned} 3(x^2 + y^2 + z^2) &= 1 \\ x^2y^2 + y^2z^2 + z^2x^2 &= xyz(x + y + z)^3. \end{aligned}$$

3 In a tetrahedron $ABCD$, the medians of the faces ABD, ACD, BCD from D make equal angles with the corresponding edges AB, AC, BC. Prove that each of these faces has area less than or equal to the sum of the areas of the other two faces.

Equivalent version of the problem: $ABCD$ is a tetrahedron. DE, DF, DG are medians of triangles DBC, DCA, DAB. The angles between DE and BC, between DF and CA, and between DG and AB are equal. Show that: area $DBC \leq$ area DCA + area DAB.

Day 2

1 The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = 0$, $a_n = a_{[n/2]} + (-1)^{n(n+1)/2}$. Show that for any positive integer k we can find n in the range $2^k \leq n < 2^{k+1}$ such that $a_n = 0$.

2 $ABCDE$ is a convex pentagon such that $DC = DE$ and $\angle C = \angle E = 90^{\cdot}$. F is a point on the side AB such that $\frac{AF}{BF} = \frac{AE}{BC}$. Show that $\angle FCE = \angle ADE$ and $\angle FEC = \angle BDC$.

3 Given any n points on a unit circle show that at most $\frac{n^2}{3}$ of the segments joining two points have length $> \sqrt{2}$.