Finals 1998

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Day 1

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1 Find all solutions in positive integers to:

$$a + b + c = xyz$$
$$x + y + z = abc$$

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2 F_n is the Fibonacci sequence $F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. Find all pairs $m > k \geq 0$ such that the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = \frac{F_k}{F_m}$ and $x_{n+1} = \frac{2x_n - 1}{1 - x_n}$ for $x_n \neq 1$, or 1 if $x_n = 1$, contains the number 1

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3 $PABCDE$ is a pyramid with $ABCDE$ a convex pentagon. A plane meets the edges PA, PB, PC, PD, PE in points A', B', C', D', E' distinct from A, B, C, D, E and P. For each of the quadrilaterals $ABB'A', BCC'B, CDD'C', DEE'D', EAA'E'$ take the intersection of the diagonals. Show that the five intersections are coplanar.

Day 2

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1 Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, $a_n = a_{n-1} + a_{[n/2]}$. Does the sequence contain infinitely many multiples of 7?

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2 The points D, E on the side AB of the triangle ABC are such that $\frac{AD}{DB}\frac{AE}{EB} = \left(\frac{AC}{CB}\right)^2$. Show that $\angle ACD = \angle BCE$.

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3 S is a board containing all unit squares in the xy plane whose vertices have integer coordinates and which lie entirely inside the circle $x^2 + y^2 = 1998^2$. In each square of S is written $+1$. An allowed move is to change the sign of every square in S in a given row, column or diagonal. Can we end up with exactly one -1 and $+1$ on the rest squares by a sequence of allowed moves?
