Finals 1999

www.artofproblemsolving.com/community/c4693

by Megus, harazi, grobber, Maverick

Day 1

1 Let D be a point on the side BC of a triangle ABC such that $AD > BC$. Let E be a point on the side AC such that $\frac{AE}{EC} = \frac{BD}{AD-BC}$. Show that $AD > BE$.

2 Given 101 distinct non-negative integers less than 5050 show that one can choose four a, b, c, d such that $a+b-c-d$ is a multiple of 5050

3 Show that one can find 50 distinct positive integers such that the sum of each number and its digits is the same.

Day 2

1 For which n do the equations have a solution in integers:

$$\begin{array}{rcl} x_1^2+x_2^2+50 & = & 16x_1+12x_2 \\ x_2^2+x_3^2+50 & = & 16x_2+12x_3 \\ \cdots\quad\cdots\quad\cdots & \cdots & \cdots\quad\cdots \\ x_{n-1}^2+x_n^2+50 & = & 16x_{n-1}+12x_n \\ x_n^2+x_1^2+50 & = & 16x_n+12x_1 \end{array}$$

2 Prove that for any $2n$ real numbers $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$, we have $\sum_{i<j}|a_i-a_j| + \sum_{i<j}|b_i-b_j| \leq \sum_{i,j\in[1,n]}|a_i-b_j|$.

3 Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA}\cdot\frac{AE}{EF}\cdot\frac{FD}{DB} = 1.$$