Finals 2001

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Day 1

1 Prove the following inequality:
$x_1 + 2x_2 + 3x_3 + ... + nx_n \leq \frac{n(n-1)}{2} + x_1 + x_2^2 + x_3^3 + ... + x_n^n$

where $\forall_{x_i} x_i > 0$

2 Given a regular tetrahedron $ABCD$ with edge length 1 and a point P inside it.
What is the maximum value of $|PA| + |PB| + |PC| + |PD|$.

3 A sequence $x_0 = A$ and $x_1 = B$ and $x_{n+2} = x_{n+1} + x_n$ is called a Fibonacci type sequence.
Call a number C a repeated value if $x_t = x_s = c$ for t different from s.
Prove one can choose A and B to have as many repeated value as one likes but never infinite.

Day 2

1 Assume that a, b are integers and n is a natural number. $2^n a + b$ is a perfect square for every n.Prove that $a = 0$.

2 Let $ABCD$ be a parallelogram and let K and L be points on the segments BC and CD, respectively, such that $BK \cdot AD = DL \cdot AB$. Let the lines DK and BL intersect at P. Show that $\measuredangle DAP = \measuredangle BAC$.

3 Given positive integers $n_1 < n_2 < ... < n_{2000} < 10^{100}$. Prove that we can choose from the set $\{n_1, ..., n_{2000}\}$ nonempty, disjont sets A and B which have the same number of elements, the same sum and the same sum of squares.