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Day 1

1 Find all the natural numbers a, b, c such that:

1) a^2+1 and b^2+1 are primes
2) $(a^2+1)(b^2+1)=(c^2+1)$

2 On sides AC and BC of acute-angled triangle ABC rectangles with equal areas $ACPQ$ and $BKLC$ were built exterior. Prove that midpoint of PL, point C and center of circumcircle are collinear.

3 Three non-negative integers are written on a blackboard. A move is to replace two of the integers k, m by $k+m$ and $|k-m|$. Determine whether we can always end with triplet which has at least two zeros

Day 2

1 $x_1, ..., x_n$ are non-negative reals and $n \geq 3$. Prove that at least one of the following inequalities is true:

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \geq \frac{n}{2},$$

$$\sum_{i=1}^{n} \frac{x_i}{x_{i-1}+x_{i-2}} \geq \frac{n}{2}.$$

2 There is given a triangle ABC in a space. A sphere does not intersect the plane of ABC. There are 4 points K, L, M, P on the sphere such that AK, BL, CM are tangent to the sphere and $\frac{AK}{AP} = \frac{BL}{BP} = \frac{CM}{CP}$. Show that the sphere touches the circumsphere of $ABCP$.

3 k is a positive integer. The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = k+1$, $a_{n+1} = a_n^2 - ka_n + k$. Show that a_m and a_n are coprime (for $m \neq n$).