Finals 2003

www.artofproblemsolving.com/community/c4697

by Amir Hossein, 77ant

1 In an acute-angled triangle ABC, CD is the altitude. A line through the midpoint M of side AB meets the rays CA and CB at K and L respectively such that $CK = CL$. Point S is the circumcenter of the triangle CKL. Prove that $SD = SM$.

---

2 Let $0 < a < 1$ be a real number. Prove that for all finite, strictly increasing sequences $k_1, k_2, \dots, k_n$ of non-negative integers we have the inequality

$$\left(\sum_{i=1}^{n} a^{k_i}\right)^2 < \frac{1+a}{1-a}\sum_{i=1}^{n} a^{2k_i}.$$

---

3 Find all polynomials W with integer coefficients satisfying the following condition: For every natural number $n, 2^n - 1$ is divisible by $W(n)$.

---

4 A prime number p and integers x, y, z with $0 < x < y < z < p$ are given. Show that if the numbers x^3, y^3, z^3 give the same remainder when divided by p, then $x^2 + y^2 + z^2$ is divisible by $x + y + z$.

---

5 The sphere inscribed in a tetrahedron $ABCD$ touches face ABC at point H. Another sphere touches face ABC at O and the planes containing the other three faces at points exterior to the faces. Prove that if O is the circumcenter of triangle ABC, then H is the orthocenter of that triangle.

---

6 Let n be an even positive integer. Show that there exists a permutation $(x_1, x_2, \dots, x_n)$ of the set $\{1, 2, \dots, n\}$, such that for each $i \in \{1, 2, \dots, n\}$, x_{i+1} is one of the numbers $2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1$, where $x_{n+1} = x_1$.
