Finals 2004

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by mr.danh

Day 1

1 A point D is taken on the side AB of a triangle ABC. Two circles passing through D and touching AC and BC at A and B respectively intersect again at point E. Let F be the point symmetric to C with respect to the perpendicular bisector of AB. Prove that the points D, E, F lie on a line.

2 Let P be a polynomial with integer coefficients such that there are two distinct integers at which P takes coprime values. Show that there exists an innite set of integers, such that the values P takes at them are pairwise coprime.

3 On a tournament with $n \geq 3$ participants, every two participants played exactly one match and there were no draws. A three-element set of participants is called a *draw-triple* if they can be enumerated so that the rst defeated the second, the second defeated the third, and the third defeated the rst. Determine the largest possible number of draw-triples on such a tournament.

Day 2

4 Let real numbers a, b, c. Prove that $\sqrt{2(a^2+b^2)}+\sqrt{2(b^2+c^2)}+\sqrt{2(c^2+a^2)} \geq \sqrt{3(a+b)^2+3(b+c)^2+3(c$

5 Find the greatest possible number of lines in space that all pass through a single point and the angle between any two of them is the same.

6 An integer $m > 1$ is given. The innite sequence $(x_n)_{n\geq 0}$ is dened by $x_i = 2^i$ for $i < m$ and $x_i = x_{i-1} + x_{i-2} + \cdots + x_{i-m}$ for $i \geq m$.
Find the greatest natural number k such that there exist k successive terms of this sequence which are divisible by m.